Is $\{a + b\sqrt{2} : a, b \in \mathbb{Z}\}$ Dense in $\mathbb{R}$?

Setting

Dense in $\mathbb{R}$

Def Let $\mathbb{D} \subseteq \mathbb{R}$. $\mathbb{D}$ is **dense in** $\mathbb{R}$ if

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3. $\mathbb{N}$ and $\mathbb{Z}$ are not dense in $\mathbb{R}$.

More Domains

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Answer on next slide.

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3. More generally, if $\gamma \in \mathbb{I}$ then

$$\{a + b\gamma : a, b \in \mathbb{Z}\}$$

   is dense in $\mathbb{R}$

Theorems About $\mathbb{D} = \{a + b\sqrt{2}\}$

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Thm If $r_1, r_2 \in \mathbb{R}^{>0}$ and $r_1 < r_2$ then

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Numbers in $\mathbb{D}$ Can Be Small

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$\mathrm{H}(\pi) = 0.14159\ldots$.

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$\mathrm{H}(4) = 0$.

Numbers in $\mathbb{D}$ Can Be Small (cont)

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Take the numbers between 0 and 1 and partition them into $\left(0, \frac{1}{n}\right]$, $\left(\frac{1}{n}, \frac{2}{n}\right]$, $\ldots$, $\left(\frac{n-1}{n}, 1\right]$

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Take the numbers between 0 and 1 and partition them into $\left(0, \frac{1}{n}\right]$, $\left(\frac{1}{n}, \frac{2}{n}\right]$, ..., $\left(\frac{n-1}{n}, 1\right]$

Map the set $\{1, \ldots, n\} \times \{1, \ldots, n\}$ into those intervals.

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$(0, 0.25]$, $(0.25, 0.5]$, $(0.5, 0.75]$, $(0.75, 1]$.

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We show where a few of the ordered pairs go.

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$(4, 1)$: $4 + 1 \times \sqrt{2} = 5.414$. $\mathrm{H}(4.414) = 0.414 \rightarrow (0.25, 0.5]$.

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Numbers in $\mathbb{D}$ Can Be Small (cont)

Take the numbers between 0 and 1 and partition them into
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$(2, 3)$: $2 + 3 \times \sqrt{2} = 6.242$. $\mathrm{H}(6.242) = 0.242 \rightarrow (0, 0.25]$.

Numbers in $\mathbb{D}$ Can Be Small (cont)

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$(2, 3)$: $2 + 3 \times \sqrt{2} = 6.242$. $\mathrm{H}(6.242) = 0.242 \rightarrow (0, 0.25]$.
$(1, 4)$: $1 + 4 \times \sqrt{2} = 6.656$. $\mathrm{H}(6.656) = 0.656. \rightarrow (0.5, 0.75]$.

Numbers in $\mathbb{D}$ Can Be Small (cont)

In the last slide we described a function from $\{1, \ldots, n\} \times \{1, \ldots, n\}$ to a set of n intervals.

Numbers in $\mathbb{D}$ Can Be Small (cont)

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Since $n < n^2$, **by the Pigeonhole Principle** there exists 2 ordered pairs that map to the same interval.
(Actually there exists more but we do not need that.)

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(Actually there exists more but we do not need that.)

Let (a, b) and (c, d) be two different ordered pairs that map to the same interval.

Numbers in $\mathbb{D}$ Can Be Small (cont)

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Numbers in $\mathbb{D}$ Can Be Small (cont)

(a, b) and (c, d) map to the same interval.

So $\mathrm{H}(a + b\sqrt{2})$ and $\mathrm{H}(c + d\sqrt{2})$ are within $\frac{1}{n}$ of each other.

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We can assume $a + b\sqrt{2} - e < c + d\sqrt{2} - f$.

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SO

1. $(c + d\sqrt{2} - f) - (a + b\sqrt{2} - e) < \frac{1}{n}$ since in same interval.

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We can assume $a + b\sqrt{2} - e < c + d\sqrt{2} - f$.
SO

1. $(c + d\sqrt{2} - f) - (a + b\sqrt{2} - e) < \frac{1}{n}$ since in same interval.
2. $(c + d\sqrt{2} - f) - (a + b\sqrt{2} - e) > 0$ since if not then $\sqrt{2} \in \mathbb{Q}$.

Numbers in $\mathbb{D}$ Can Be Small (cont)

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SO

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So
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Numbers in $\mathbb{D}$ Can Be Small (cont)

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So

$(x, y) = (c + e - f - a, b - d) \in \mathbb{Z} \times \mathbb{Z}$ works.

$\mathbb{D}$ is Dense in $\mathbb{R}^{>0}$

Thm If $r_1, r_2 \in \mathbb{R}^{>0}$ and $r_1 < r_2$ then

$$(\exists a, b \in \mathbb{Z})[r_1 < a + b\sqrt{2} < r_2].$$

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Let n be such $\frac{1}{n} < \min\{r_2 - r_1, r_1\}$.

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Continued on next page.

$\mathbb{D}$ is Dense in $\mathbb{R}^{>0}$ (cont)

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[	|	(	)
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$(a, b) = ((m+1)x, (m+1)y)$ works.

Where Did This Come From?

The Origin of the Question

I though of the question

$$\text{is } \{\mathbf{a} + \mathbf{b}\sqrt{\mathbf{2}} : \mathbf{a}, \mathbf{b} \in \mathbb{Z}\} \text{ dense?}$$

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What about the proof?

All of the ideas for the proof were known but in a different context.
It comes from Dirichlets' Theorem on Approximationg Irrationals.
We won't be doing that.

Approximating $\gamma \in \mathbb{I}$ with Rationals

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We prove Dirichlet's Theorem on approximations of irrationals by rationals. You are already familiar with most of the ideas.

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Take the numbers between 0 and 1 and partition them into
$\left(0, \frac{1}{n^2}\right]$, $\left(\frac{1}{n^2}, \frac{2}{n^2}\right]$, ..., $\left(\frac{n^2-1}{n^2}, 1\right]$

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Map the set $\{1, \ldots, n+1\} \times \{1, \ldots, n+1\}$ into those intervals.

Map (a, b) to the interval that $\mathrm{H}(a + b\gamma)$ is in.

Approximating $\gamma \in \mathbb{I}$ with Rationals

In the last slide we described a function from $\{1, \ldots, n+1\} \times \{1, \ldots, n+1\}$ to a set of n^2 intervals.

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Let (a, b) and (c, d) be two different ordered pairs that map to the same interval.

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Can the Approximation Theorem Be Improved?

Dirichlet Proved:
Thm Let $\gamma \in \mathbb{I}$. $(\forall n \in \mathbb{N})(\exists m \in \mathbb{Z})\left[|\gamma - \frac{m}{n}| < \frac{1}{n^2}\right]$.

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3. $\frac{\sqrt{5}+1}{2}$ is one of those γ.

Questions Left

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1. Is the proof that $\{a + b\sqrt{2} : a, b \in \mathbb{Z}\}$ is dense easy enough to do in the ordinary CMSC 250?

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   Emily says its because I look at things more pedagogically.