Grid Colorings that Avoid Rectangles

October 11, 2024

Credit Where Credit is Due

This talk is based on a paper by
Stephen Fenner
William Gasarch
Charles Glover
Semmy Purewal

Ramsey Theory

This talk is an example of Ramsey Theory.

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I will be teaching CMSC 752:
Ramsey Theory and its “Applications”
in the Spring of 2025.

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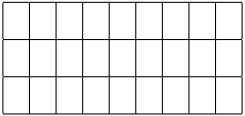
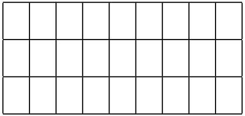
So this talk is an advertisement for the course.

2-Coloring 3×9

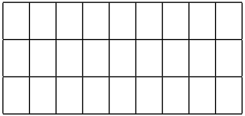


2-Coloring 3×9



Is there a 2-coloring of 3×9 with no mono rectangles?

2-Coloring 3×9



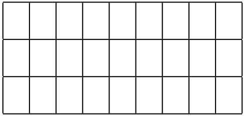
Is there a 2-coloring of 3×9 with no mono rectangles?
What is a mono rectangle? Here is a an example:

2-Coloring 3×9

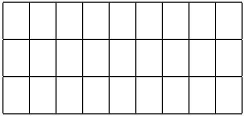
Is there a 2-coloring of 3×9 with no mono rectangles?
What is a mono rectangle? Here is a an example:

	R					R		
	R					R		

2-Coloring 3×9: Vote

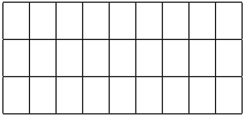


2-Coloring 3×9: Vote



Vote

2-Coloring 3×9: Vote



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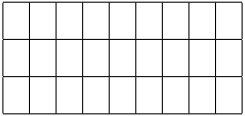
1. There is a 2-coloring of 3×9 with NO mono rectangles.

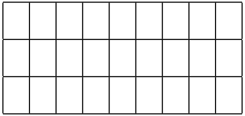
2-Coloring 3×9: Vote



Vote

1. There is a 2-coloring of 3×9 with NO mono rectangles.
2. All 2-colorings of 3×9 have a mono rectangle.

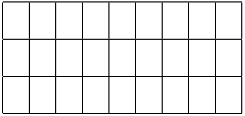
2-Coloring 3×9: Vote



Vote

1. There is a 2-coloring of 3×9 with NO mono rectangles.
2. All 2-colorings of 3×9 have a mono rectangle.
3. The problem is **UNKNOWN TO SCIENCE**.

2-Coloring 3×9: Vote



Vote

1. There is a 2-coloring of 3×9 with NO mono rectangles.
2. All 2-colorings of 3×9 have a mono rectangle.
3. The problem is **UNKNOWN TO SCIENCE**.

Answer on the next slide.

All 2-colorings of 3×9 have a mono rectangle

Given a 2-coloring of 3×9 look at each column.

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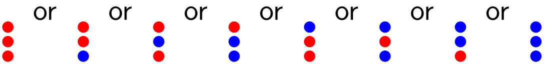
Each column is either



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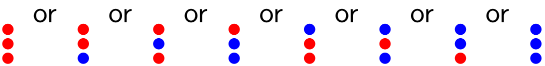

Key: *A 2-coloring of* 3×9 *is an 8-coloring of the 9 columns.*

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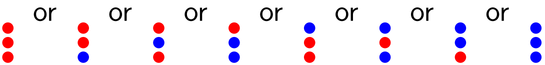

Key: *A 2-coloring of* 3×9 *is an 8-coloring of the 9 columns.*

So some column-color appears twice.

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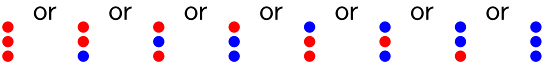


Key: *A 2-coloring of* 3×9 *is an 8-coloring of the 9 columns.*

So some column-color appears twice.

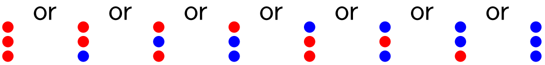
Example:

	R					R		
	B					B		
	R					R		

All 2-colorings of 3×9 have a mono rectangle

Given a 2-coloring of 3×9 look at each column.

Each column is either



Key: *A 2-coloring of* 3×9 *is an 8-coloring of the 9 columns.*

So some column-color appears twice.
Example:

	R					**R**		
	B					**B**		
	R					**R**		

Can easily show that the two repeat-columns lead to a mono rectangle.

2-Coloring 3×8, 3×7, ...

Work in groups:

2-Coloring 3×8, 3×7, ...

Work in groups:

1. Is there a 2-coloring of 3×8 with no mono rectangles?

2-Coloring 3×8, 3×7, ...

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2-Coloring 3×8, 3×7, ...

Work in groups:

1. Is there a 2-coloring of 3×8 with no mono rectangles?
2. Is there a 2-coloring of 3×7 with no mono rectangles?
3. Is there a 2-coloring of 3×6 with no mono rectangles?

2-Coloring 3×8, 3×7, ...

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2. Is there a 2-coloring of 3×7 with no mono rectangles?
3. Is there a 2-coloring of 3×6 with no mono rectangles?
4. Is there a 2-coloring of 3×5 with no mono rectangles?

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4. Is there a 2-coloring of 3×5 with no mono rectangles?
5. Is there a 2-coloring of 3×4 with no mono rectangles?

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3. Is there a 2-coloring of 3×6 with no mono rectangles?
4. Is there a 2-coloring of 3×5 with no mono rectangles?
5. Is there a 2-coloring of 3×4 with no mono rectangles?
6. Is there a 2-coloring of 3×3 with no mono rectangles? YES:

2-Coloring 3×8, 3×7, ...

Work in groups:

1. Is there a 2-coloring of 3×8 with no mono rectangles?
2. Is there a 2-coloring of 3×7 with no mono rectangles?
3. Is there a 2-coloring of 3×6 with no mono rectangles?
4. Is there a 2-coloring of 3×5 with no mono rectangles?
5. Is there a 2-coloring of 3×4 with no mono rectangles?
6. Is there a 2-coloring of 3×3 with no mono rectangles? YES:

Example:

R	B	R
R	B	B
R	R	B

2-Coloring 3×8, 3×7, ...

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1. Is there a 2-coloring of 3×8 with no mono rectangles?

2-Coloring 3×8, 3×7, ...

1. Is there a 2-coloring of 3×8 with no mono rectangles?
   NO: to avoid a repeat col must have col Easily get mono rectangle.

2-Coloring 3×8, 3×7, ...

1. Is there a 2-coloring of 3×8 with no mono rectangles?
   NO: to avoid a repeat col must have col  Easily get mono rectangle.
2. Is there a 2-coloring of 3×7 with no mono rectangles?

2-Coloring 3×8, 3×7, ...

1. Is there a 2-coloring of 3×8 with no mono rectangles?
   NO: to avoid a repeat col must have col Easily get mono rectangle.
2. Is there a 2-coloring of 3×7 with no mono rectangles?
   NO: to avoid a repeat col must have col OR
   Easily get mono rectangle.

2-Coloring 3×8, 3×7, ...

1. Is there a 2-coloring of 3×8 with no mono rectangles?
   NO: to avoid a repeat col must have col Easily get mono rectangle.
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3. Is there a 2-coloring of 3×6 with no mono rectangles?

2-Coloring 3×8, 3×7, ...

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   NO: to avoid a repeat col must have col  OR 
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3. Is there a 2-coloring of 3×6 with no mono rectangles?
   YES

2-Coloring 3×8, 3×7, ...

1. Is there a 2-coloring of 3×8 with no mono rectangles?
   NO: to avoid a repeat col must have col $\begin{matrix} \color{red}\bullet \\ \color{red}\bullet \\ \color{red}\bullet \end{matrix}$ Easily get mono rectangle.
2. Is there a 2-coloring of 3×7 with no mono rectangles?
   NO: to avoid a repeat col must have col $\begin{matrix} \color{red}\bullet \\ \color{red}\bullet \\ \color{red}\bullet \end{matrix}$ OR $\begin{matrix} \color{blue}\bullet \\ \color{blue}\bullet \\ \color{blue}\bullet \end{matrix}$
   Easily get mono rectangle.
3. Is there a 2-coloring of 3×6 with no mono rectangles?
   YES

R	R	R	B	B	B
R	B	B	R	B	R
B	R	B	B	R	R

2-Coloring 3×8, 3×7, ...

1. Is there a 2-coloring of 3×8 with no mono rectangles?
   NO: to avoid a repeat col must have col $\color{red}\begin{matrix}\bullet\\\bullet\\\bullet\end{matrix}$ Easily get mono rectangle.
2. Is there a 2-coloring of 3×7 with no mono rectangles?
   NO: to avoid a repeat col must have col $\color{red}\begin{matrix}\bullet\\\bullet\\\bullet\end{matrix}$ OR $\color{blue}\begin{matrix}\bullet\\\bullet\\\bullet\end{matrix}$
   Easily get mono rectangle.
3. Is there a 2-coloring of 3×6 with no mono rectangles?
   YES

R	R	R	B	B	B
R	B	B	R	B	R
B	R	B	B	R	R

4. Hence there is a 2-coloring of 3×5, 3×4, 3×3 with no mono rectangles.

What Do We Know?

$a \times b$ is *2-colorable* if there is a 2-coloring with no mono rectangles.

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- $3 \times b$ where $b \geq 7$ NOT 2-colorable.

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- 4×4, 4×5, 4×6 unknown so far.

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- 5×5, 5×6 unknown so far.

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- 5×5, 5×6 unknown so far.
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- 5×5, 5×6 unknown so far.
- $5 \times b$ where $b \geq 7$ NOT 2-colorable.
- 6×6 unknown so far.
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Work on the 4×4, 4×5 4×6.

4×6 IS 2-Colorable

R	R	R	B	B	B
R	B	B	R	R	B
B	R	B	R	B	R
B	B	R	B	R	R

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Work on 5×5, 5×6.

5×5 IS NOT 2-Colorable!

Let COL be a 2-coloring of 5×5.

5×5 IS NOT 2-Colorable!

Let COL be a 2-coloring of 5×5.
Some color must occur ≥ 13 times.

Case 1: There is a column with 5 R's

Case 1: There is a column with 5 **R**'s.

$$\begin{array}{ccccc} \mathbf{R} & \circ & \circ & \circ & \circ \\ \mathbf{R} & \circ & \circ & \circ & \circ \\ \mathbf{R} & \circ & \circ & \circ & \circ \\ \mathbf{R} & \circ & \circ & \circ & \circ \\ \mathbf{R} & \circ & \circ & \circ & \circ \end{array}$$

Remaining columns have ≤ 1 **R** so

$$\text{Number of } \mathbf{R}\text{'s } \leq 5 + 1 + 1 + 1 + 1 = 9 < 13.$$

Case 2: There is a column with 4 R's

Case 2: There is a column with 4 R's.

$$
\begin{array}{ccccc}
\mathbf{R} & \circ & \circ & \circ & \circ \\
\mathbf{R} & \circ & \circ & \circ & \circ \\
\mathbf{R} & \circ & \circ & \circ & \circ \\
\mathbf{R} & \circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ & \circ
\end{array}
$$

Remaining columns have ≤ 2 **R**'s

Number of **R**'s $\leq 4 + 2 + 2 + 2 + 2 \leq 12 < 13$

Case 3: Max in a column is 3 R's

Case 3: Max in a column is 3 **R**'s.
Case 3a: There are ≤ 2 columns with 3 **R**'s.

$$\text{Number of } \mathbf{R}\text{'s} \le 3+3+2+2+2 \le 12 < 13.$$

Case 3b: There are ≥ 3 columns with 3 **R**$'$s.

$$\begin{array}{ccccc}
\mathbf{R} & \circ & \circ & \circ & \circ \\
\mathbf{R} & \circ & \circ & \circ & \circ \\
\mathbf{R} & \mathbf{R} & \circ & \circ & \circ \\
\circ & \mathbf{R} & \circ & \circ & \circ \\
\circ & \mathbf{R} & \circ & \circ & \circ
\end{array}$$

Can't put in a third column with 3 **R**'s!

Case 4: Max in a column is $\leq 2R$'s

Case 4: Max in a column is ≤ 2**R**'s.

$$\text{Number of } \mathbf{R}\text{'s} \leq 2+2+2+2+2 \leq 10 < 13.$$

No more cases. We are Done! Q.E.D.

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- $4 \times b$ where $b \geq 7$ NOT 2-colorable.

What Do We Know?

What we know

- $2 \times b$ is always 2-colorable
- 3×3, 3×4, 3×5, 3×6 are all 2-colorable.
- $3 \times b$ where $b \geq 7$ NOT 2-colorable.
- 4×4, 4×5, 4×6 are 2-colorable
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- 5×5, 5×6 NOT 2-colorable.
- $5 \times b$ where $b \geq 7$ NOT 2-colorable.
- 6×6 NOT 2-colorable.

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We now know **exactly** what grids are 2-colorable.

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What we know

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We now know **exactly** what grids are 2-colorable.
Can we say it more succinctly?

Obstruction Sets

Def $n \times m$ contains $a \times b$ if $a \leq n$ and $b \leq m$.
Thm For all c there exists a unique finite set of grids OBS_c such that

$n \times m$ is c-colorable **iff**
$n \times m$ does not contain any element of OBS_c.

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1. $\mathrm{OBS}_2 = \{3 \times 7, 7 \times 3, 5 \times 5\}$.
2. Can prove Thm using well-quasi-orderings. No bound on $|\mathrm{OBS}_c|$.
3. We showed $2\sqrt{c}(1 - o(1)) \leq |\mathrm{OBS}_c| \leq 2c^2$.

Research Question

The theorem

> **$a \times b$ is 2-colorable iff no elements of $\mathbf{OBS_2 = \{3 \times 7, 7 \times 3, 5 \times 5\}}$ fits into $a \times b$**

was proven by cleverness.

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3-coloring is known. 4-coloring is known. 5-coloring is open!

Main Question

Fix c

What is OBS_c

Main Question

Fix c

What is OBS_c

We developed tools to get us both colorings and non-colorings. They helped us get some of our results, but (alas) to many had to be done ad-hoc.

3-COLORABILITY

We will EXACTLY Characterize which $n \times m$ are 3-colorable!

Easy 3-Colorable Results

Thm

1. The following grids **are not** 3-colorable.

Easy 3-Colorable Results

Thm

1. The following grids **are not** 3-colorable.
   4×19, 19×4, 5×16, 16×5, 7×13, 13×7, 10×12, 12×10, 11×11.

Easy 3-Colorable Results

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1. The following grids **are not** 3-colorable.
   4×19, 19×4, 5×16, 16×5, 7×13, 13×7, 10×12, 12×10, 11×11.
2. The following grids **are** 3-colorable.

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Thm

1. The following grids **are not** 3-colorable.
   4×19, 19×4, 5×16, 16×5, 7×13, 13×7, 10×12, 12×10, 11×11.
2. The following grids **are** 3-colorable.
   3×19, 19×3, 4×18, 18, 6×15, 15×6, 9×12, 12×9.

Easy 3-Colorable Results

Thm

1. The following grids **are not** 3-colorable.
   4×19, 19×4, 5×16, 16×5, 7×13, 13×7, 10×12, 12×10, 11×11.
2. The following grids **are** 3-colorable.
   3×19, 19×3, 4×18, 18, 6×15, 15×6, 9×12, 12×9.

Follows from tools.

10×10 is 3-colorable

Thm 10×10 is 3-colorable.
UGLY! TOOLS DID NOT HELP AT ALL!!

R	R	R	R	B	B	G	G	B	G
R	B	B	G	R	R	R	G	G	B
G	R	B	G	R	B	B	R	R	G
G	B	R	B	B	R	G	R	G	R
R	B	G	G	G	B	G	B	R	R
G	R	B	B	G	G	R	B	B	R
B	G	R	B	G	B	R	G	R	B
B	B	G	R	R	G	B	G	B	R
G	G	G	R	B	R	B	B	R	B
B	G	B	R	B	G	R	R	G	G

10×11 is not 3-colorable

Thm 10×11 is not 3-colorable.
You don't want to see this. UGLY case hacking.

Complete Char of 3-colorability

Thm $\mathrm{OBS}_3 =$

$$\{4 \times 19, 5 \times 16, 7 \times 13, 10 \times 11, 11 \times 10, 13 \times 7, 16 \times 5, 19 \times 4\}$$

Complete Char of 3-colorability

Thm $\mathrm{OBS}_3 =$

$$\{4 \times 19, 5 \times 16, 7 \times 13, 10 \times 11, 11 \times 10, 13 \times 7, 16 \times 5, 19 \times 4\}$$

Follows from our tools and the ad-hoc results.

Research Question

The theorem

> **$a \times b$ is 3-colorable iff no elements of $\mathbf{OBS_3} =$ $\{\mathbf{4 \times 19, 5 \times 16, 7 \times 13, 10 \times 11}\} \cup$ $\{\mathbf{11 \times 10, 13 \times 7, 16 \times 5, 19 \times 4}\}$ fits into $a \times b$**

was proven by cleverness.

Research Question

The theorem

> $a \times b$ **is 3-colorable iff no elements of**
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Research Question

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Since 2-coloring has been solved why ask this question?
4-coloring is known. 5-coloring is open!
If the answer is NO then we have found a problem that AI can't do!

4-COLORABILITY

From now on $G_{a,b}$ is $a \times b$.
We will EXACTLY Characterize which $G_{n,m}$ are 4-colorable!

Easy NOT 4-Colorable Results

Thm The following grids **are** NOT 4-colorable:

1. $G_{5,41}$ and $G_{41,5}$
2. $G_{6,31}$ and $G_{31,6}$
3. $G_{7,29}$ and $G_{29,7}$
4. $G_{9,25}$ and $G_{25,9}$
5. $G_{10,23}$ and $G_{23,10}$
6. $G_{11,22}$ and $G_{22,11}$
7. $G_{13,21}$ and $G_{21,13}$
8. $G_{17,20}$ and $G_{20,17}$
9. $G_{18,19}$ and $G_{19,18}$

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4. $G_{9,25}$ and $G_{25,9}$
5. $G_{10,23}$ and $G_{23,10}$
6. $G_{11,22}$ and $G_{22,11}$
7. $G_{13,21}$ and $G_{21,13}$
8. $G_{17,20}$ and $G_{20,17}$
9. $G_{18,19}$ and $G_{19,18}$

Follows from our tools.

Easy IS 4-Colorable Results

Thm The following grids **are** 4-colorable:

1. $G_{4,41}$ and $G_{41,4}$.
2. $G_{5,40}$ and $G_{40,5}$.
3. $G_{6,30}$ and $G_{30,6}$.
4. $G_{8,28}$ and $G_{28,8}$.
5. $G_{16,20}$ and $G_{20,16}$.

Easy IS 4-Colorable Results

Thm The following grids **are** 4-colorable:

1. $G_{4,41}$ and $G_{41,4}$.
2. $G_{5,40}$ and $G_{40,5}$.
3. $G_{6,30}$ and $G_{30,6}$.
4. $G_{8,28}$ and $G_{28,8}$.
5. $G_{16,20}$ and $G_{20,16}$.

Follows from our tools.

Theorems with UGLY Proofs

Thm

1. $G_{17,19}$ **is** NOT 4-colorable: Some Tools, Some ad-hoc.

Theorems with UGLY Proofs

Thm

1. $G_{17,19}$ **is** NOT 4-colorable: Some Tools, Some ad-hoc.
2. $G_{24,9}$ **is** 4-colorable: Some Tools, Some ad-hoc.

Results about 4-COL So Far in This Talk

Thm

1. The following grids are in OBS_4:
$G_{5,41}$, $G_{6,31}$, $G_{7,29}$, $G_{9,25}$, $G_{10,23}$, $G_{11,22}$,
$G_{22,11}$, $G_{23,10}$, $G_{25,9}$, $G_{29,7}$, $G_{31,6}$, $G_{41,5}$.

Results about 4-COL So Far in This Talk

Thm

1. The following grids are in OBS_4:
   $G_{5,41}$, $G_{6,31}$, $G_{7,29}$, $G_{9,25}$, $G_{10,23}$, $G_{11,22}$,
   $G_{22,11}$, $G_{23,10}$, $G_{25,9}$, $G_{29,7}$, $G_{31,6}$, $G_{41,5}$.
2. The following grids status is unknown:
   $G_{10,22}$, $G_{12,21}$, $G_{17,17}$, $G_{17,18}$, $G_{18,18}$,
   $G_{18,17}$, $G_{21,21}$, $G_{22,10}$.

Rectangle Free Conjecture

The following is obvious:

Lemma Let $n, m, c \in \mathbb{N}$. If $G_{n,m}$ is c-colorable then some color occurs $\geq \lceil nm/c \rceil$ times. Hence there is a rectangle free subset of $G_{n,m}$ with $\geq \lceil nm/c \rceil$ elements.

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Lemma Let $n, m, c \in \mathrm{N}$. If $G_{n,m}$ is c-colorable then some color occurs $\geq \lceil nm/c \rceil$ times. Hence there is a rectangle free subset of $G_{n,m}$ with $\geq \lceil nm/c \rceil$ elements.

Rectangle-Free Conjecture (RFC) is the converse:

Let $n, m, c \geq 2$. If there is a rectangle free subset of size of $G_{n,m}$ which is $\geq \lceil nm/c \rceil$ then $G_{n,m}$ is c-colorable.

Rect-Free Subset of $G_{22,10}$ of size $55 = \left\lceil \frac{22 \cdot 10}{4} \right\rceil$

	01	02	03	04	05	06	07	08	09	10
1	•						•			
2		•					•			
3			•				•			
4				•			•			
5					•		•			
6						•	•			
7	•	•						•		
8			•	•				•		
9					•	•		•		
10		•	•						•	
11				•	•				•	
12	•					•			•	
13	•			•						•
14		•				•				•
15			•		•					•
16		•			•					
17	•		•							
18				•		•				
19			•			•				
20		•		•						
21	•				•					
22							•	•	•	•

4-coloring of $G_{22,10}$ Due to Brad Loren

	1	2	3	4	5	6	7	8	9	10
1	O	G	R	R	G	G	O	O	B	B
2	G	O	B	G	B	B	O	R	O	R
3	B	G	B	R	O	O	G	R	O	B
4	O	O	G	G	R	R	B	B	G	O
5	O	B	O	O	G	R	R	G	G	R
6	O	B	R	B	R	O	G	R	G	G
7	G	O	G	O	B	O	R	B	R	G
8	O	R	R	B	O	B	G	G	B	R
9	O	B	B	R	R	G	R	G	O	G
10	R	R	B	B	O	G	R	B	G	O
11	R	G	G	O	R	B	B	G	O	R
12	R	B	R	G	G	O	O	B	B	G
13	B	R	G	B	G	R	B	R	O	O
14	G	G	O	B	B	O	R	R	G	B
15	R	G	O	R	B	R	B	O	O	G
16	B	B	O	G	O	B	O	G	R	R
17	G	O	B	R	O	G	B	O	B	R
18	R	B	G	O	B	G	O	R	R	O
19	G	B	R	O	O	R	B	G	R	B
20	B	R	O	G	R	G	G	B	R	O
21	B	R	G	R	B	O	G	O	B	O
22	G	O	O	R	G	B	G	B	R	B

Rect-Free Subset of $G_{21,12}$ of size 63 $= \left\lceil \frac{21\cdot 12}{4} \right\rceil$

	01	02	03	04	05	06	07	08	09	10	11	12
1	•	•										
2	•		•									
3		•	•									
4			•	•	•							
5		•		•		•						
6	•				•	•						
7						•	•	•				
8					•		•		•			
9				•				•	•			
10						•				•	•	
11					•					•		•
12				•							•	•
13			•			•			•			•
14			•					•		•		
15			•				•				•	
16		•							•	•		
17		•			•			•			•	
18		•					•					•
19	•								•		•	
20	•							•				•
21	•			•			•			•		

Tom Sirgedas's 4-coloring of 21×12

R	B	B	G	B	R	G	O	O	G	R	G
B	R	B	R	G	B	O	G	O	G	G	R
B	B	R	B	R	G	O	O	G	R	G	G
G	O	B	O	R	R	R	B	O	G	B	O
B	G	O	R	O	R	O	R	B	O	G	B
O	B	G	R	R	O	B	O	R	B	O	G
B	O	R	G	B	O	B	G	G	O	B	R
R	B	O	O	G	B	G	B	G	R	O	B
O	R	B	B	O	G	G	G	B	B	R	O
G	R	R	B	B	R	G	O	R	O	G	O
R	G	R	R	B	B	R	G	O	O	O	G
R	R	G	B	R	B	O	R	G	G	O	O
G	O	O	R	G	B	B	O	B	R	R	G
O	G	O	B	R	G	B	B	O	G	R	R
O	O	G	G	B	R	O	B	B	R	G	R
G	O	B	G	O	G	B	R	R	R	O	B
B	G	O	G	G	O	R	B	R	B	R	O
O	B	G	O	G	G	R	R	B	O	B	R
G	G	R	O	B	O	O	R	G	B	R	B
R	G	G	O	O	B	G	O	R	B	B	R
G	R	G	B	O	O	R	G	O	R	B	B

Rect-Free Subset of $G_{18,18}$ of size $81 = \left\lceil \frac{18 \cdot 18}{4} \right\rceil$

	01	02	03	04	05	06	07	08	09	10	11	12	13	14	15	16	17	18
1		●		●										●		●	●	
2	●	●								●	●		●					
3	●								●						●	●		●
4						●			●			●	●	●				
5		●	●			●												●
6	●			●		●	●											
7							●	●		●				●				●
8			●				●		●		●						●	
9		●			●		●					●			●			
10				●							●	●						●
11	●		●		●									●				
12			●	●				●					●		●			
13					●	●		●			●					●		
14	●							●				●					●	
15				●	●				●	●								
16						●				●					●		●	
17			●							●		●				●		
18					●								●				●	●

If RFC is true then $G_{18,18}$ is 4-colorable. NOTE: If delete 2nd column and 5th Row have 74-sized RFC of $G_{17,17}$.

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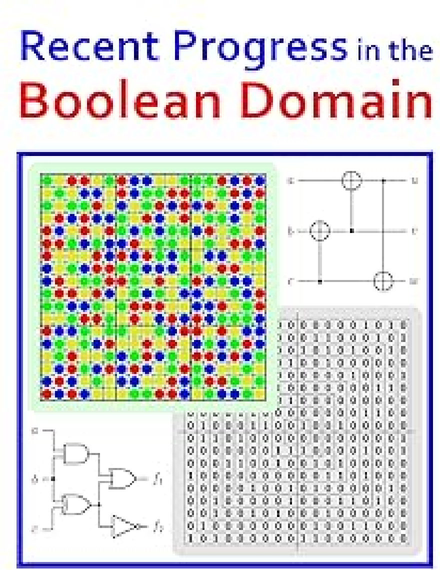
They used a very deep analysis that allowed for a strong reduction of the problem, and then used the Universal SAT-Solver Clasp.

Steinbach & Postoff's 4-Coloring of $G_{18,18}$

R	B	B	R	O	O	O	R	G	R	R	G	O	G	B	G	O	B
G	R	O	O	O	G	G	B	O	R	B	B	R	R	B	G	B	G
B	B	G	G	R	R	O	B	O	B	R	O	R	G	O	O	R	G
R	R	G	B	O	G	R	B	R	O	O	G	B	O	G	O	R	B
B	O	B	G	G	O	G	G	O	R	B	G	B	O	R	B	R	R
G	O	O	R	R	B	B	B	R	O	G	G	G	R	O	B	O	R
B	R	G	B	B	B	G	O	G	G	R	O	G	O	B	R	O	R
O	R	O	G	R	B	O	R	B	B	B	R	G	O	G	G	G	B
O	O	R	G	O	G	B	R	B	G	G	O	B	R	B	R	R	O
B	G	G	O	G	O	B	R	R	O	G	O	R	B	R	G	B	B
O	R	R	R	B	R	G	O	O	O	G	B	O	G	R	B	G	B
G	B	G	O	B	R	B	G	R	R	B	R	O	O	O	R	G	O
G	B	O	B	G	R	R	R	B	G	O	O	O	G	G	B	B	R
G	G	O	G	B	O	R	O	G	B	R	R	B	R	R	O	B	O
O	G	B	R	B	O	R	B	B	G	O	G	R	B	O	R	G	G
R	G	B	B	R	G	B	G	O	B	O	B	G	G	R	R	O	O
R	O	R	O	G	G	O	O	G	B	O	R	R	B	B	B	G	R
O	B	R	O	R	B	R	G	G	R	G	B	B	B	G	O	O	G

The 4-Coloring of 18 × 18 on Cover of a Book Edited by Steinbach



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The usual **Research Question:** Can we get OBS_4 with AI?

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If you want to see the proof that for all c, $N(c)$ exists then

Take CMSC 752 in Spring 2025

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5. Unleash AI on these problems!