The Hat Check Problem

Hat Check Problem

- n people give their hats to a hat check person.
- The hat check person gives people their hats RANDOMLY.
- What is Prob NOBODY gets their correct hat?

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1. Do you think that as n gets large the prob that nobody gets their correct hat goes **up** or goes **down** or **oscillates**?
2. It limits to a value v. Vote!
   2.1 $0 < v < \frac{1}{4}$
   2.2 $\frac{1}{4} \le v < \frac{1}{2}$.
   2.3 $\frac{1}{2} \le v < \frac{3}{4}$.
   2.4 $\frac{3}{4} \le v < 1$.

Will answer at the end of this slide packet.

$n = 2$

The two people are named $1, 2$.
The hats are labeled $1, 2$.

Number of ways the people can get their hats: $2! = 2$.
In 1 of those nobody gets their hat back.
So Prob is $\frac{1}{2}$.

Notation

P(i) will be prob that person i gets their hat back
P(i,j) will be prob that persons i and j get their hat back
etc.

ALSO
P(some) is Prob someone has the right hat.

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The three people are named $1, 2, 3$.
The hats are labeled $1, 2, 3$.

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KEY: We do prob that SOMEONE DOES get right hat.

P(1)=$\frac{1}{3}$
P(2)=$\frac{1}{3}$
P(3)=$\frac{1}{3}$
So prob that SOMEONE gets the right hat is $\frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1$.
REALLY?

$n = 3$ Continued

P(some)
$= P(1) + P(2) + P(3) - P(1,2) - P(1,3) - P(2,3) + P(1,2,3)$
$= 3P(1 \text{ right}) - 3P(1,2 \text{ right}) + P(1,2,3)$

$P(1) = \frac{1}{3}$
$P(1,2) = \frac{1}{6}$
$P(1,2,3) = \frac{1}{6}$

So Prob is $3 \times \frac{1}{3} - 3 \times \frac{1}{6} - \frac{1}{6} = \frac{2}{3}$.
So Prob NOBODY gets right hat is $1 - \frac{2}{3} = \frac{1}{3}$.

$n = 4$

P(some) = P(1) + P(2) + P(3) +P(4)
- (P(1,2)+P(1,3)+P(1,4)+P(2,3)+ P(2,4) + P(3,4))
+ (P(1,2,3)+P(1,2,4)+P(1,3,4)+P(2,3,4)
- P(1,2,3,4)

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EASIER:
P(some) = $\binom{4}{1}$P(1) - $\binom{4}{2}$P(1,2) + $\binom{4}{3}$P(1,2,3) - $\binom{4}{4}$P(1,2,3,4)

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P(1)=$\frac{3!}{4!}$
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P(1,2,3)=$\frac{1!}{4!}$
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P(some)=

$$\frac{4!}{1!3!}\frac{3!}{4!} - \frac{4!}{2!2!}\frac{2!}{4!} + \frac{4!}{3!1!}\frac{1!}{4!} - \frac{4!}{0!4!}\frac{1!}{4!}$$

$$= 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} = \frac{15}{24} = \frac{5}{8}$$

Table of Results So Far

n	Prob nobody gets their hat back
2	$\frac{1}{2} = 0.5$
3	$\frac{1}{3} = 0.33\ldots$
4	$\frac{5}{8} = 0.625$

So far it looks like it osculating, but not much evidence.

General Case

P(some) $=$
$\binom{n}{1}$P(1) - $\binom{n}{2}$P(1,2) $+$ $\binom{n}{3}$P(1,2,3) $\cdots \pm$ $\binom{n}{n}$P(1,. . .,n)

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P(1)$=\frac{(n-1)!}{n!}$
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P$(1,\ldots,n)=\frac{1!}{n!}$

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etc.

P$(1,\ldots,n)=\frac{1!}{n!}$

P(some)$=$

$$\frac{n!}{1!(n-1)!}\frac{(n-1)!}{n!}-\frac{n!}{2!(n-2)!}\frac{(n-2)!}{n!}\pm\frac{1}{n!}$$

$$=1-\frac{1}{2!}+\frac{1}{3!}-\frac{1}{4!}\pm\frac{1}{n!}$$

Pass to the Infinite Sum

We approximate

$$= 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} \pm \frac{1}{n!}$$

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Is this a good approximation? Discuss.
Yes The error term is something like

$$\frac{1}{(n+1)!} - \frac{1}{(n-1)!} + \cdots \leq \frac{1}{(n+1)!}$$

Even this upper bound is an overestimate.

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Upshot Approximation is **very good**.

How to Sum the Series

P(some)=

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So P(nobody has the right hat) =

$$\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} \cdots$$

Do you know how to sum this series?

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Do you know how to sum this series? Recall that

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\frac{1}{e} = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \cdots = \frac{1}{2!} - \frac{1}{3!} + \cdots$$

SO final answer:
Prob nobody has right hat is $\sim \frac{1}{e}$.

Reflection

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1. Are you surprised the answer is $\frac{1}{e}$?
   Surprised at how big this is?
   How small this is?
   How nice this is?
2. Are you surprised that for n large its very stable at around $\frac{1}{e}$?