Homework 02, MORALLY Due Feb 17

1. (40 points) For this problem FIRST write the program and make conjectures THEN look up whats true.

   (a) (0 points) Write a program that does the following: For every $1 \le n \le 900$:

      - Using the greedy method (Leo will tell you about that in class) determine natural numbers $(x_1, x_2, \ldots, x_k)$ such that $x_1^3 + \cdots + x_k^3 = n$. Using brute force (Leo will tell you about that in class) determine natural numbers $(x_1, x_2, \ldots, x_L)$ such that $x_1^3 + \cdots + x_L^3 = n$ with L minimum.
      - Below I have the first 9 rows of the output. We use Gr-k and Br-L so the table fits on the page.

n	Gr-k	Greedy-x_i's	Br-L	Brute-x_i's
1	1	$1 = 1^3$	1	$1 = 1^3$
2	2	$2 = 1^3 + 1$	2	$2 = 1^3 + 1^3$
3	3	$3 = 1^3 + 1^3 + 1^3$	3	$3 = 1^3 + 1^3 + 1^3$
4	4	$4 = 1^3 + 1^3 + 1^3 + 1^3$	4	$4 = 1^3 + 1^3 + 1^3 + 1^3$
5	5	$5 = 1^3 + 1^3 + 1^3 + 1^3 + 1^3$	5	$5 = 1^3 + 1^3 + 1^3 + 1^3 + 1^3$
6	6	$6 = 1^3 + 1^3 + 1^3 + 1^3 + 1^3 + 1^3$	6	$6 = 1^3 + 1^3 + 1^3 + 1^3 + 1^3 + 1^3$
7	7	$7 = 1^3 + 1^3 + 1^3 + 1^3 + 1^3 + 1^3 + 1^3$	7	$7 = 1^3 + 1^3 + 1^3 + 1^3 + 1^3 + 1^3 + 1^3$
8	1	$8 = 2^3$	1	$8 = 2^3$
9	2	$9 = 2^3 + 1^3$	2	$9 = 2^3 + 1^3$

DO NOT hand in the program or the output.

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(b) (30 points)

i. (15 points) Create a table that shows how many numbers ≤ 900 require 1 cube, 2 cubes, 3 cubes, and so on for both the Greedy and Brute Force solutions. Below is a sample table (THIS DOES NOT CONTAIN THE CORRECT ANSWER):

# of Cubes	Greedy	Brute Force
1	3	8
2	4	2
3	28	17
⋮	⋮	⋮

ii. (15 points) Based on this data make conjectures of the following forms:

A. Every n is the sum of $\leq XXX$ cubes. Write your conjecture in quantifiers.

B. All but a finite number of n is the sum of $\leq XXX$ cubes. Write your conjecture in quantifiers.

C. For large n, the Greedy algorithm gives the same answer as brute force for approximately XXX fraction of $\{1, ..., n\}$. (Note that brute force gives the *correct* answer so we are wondering how often Greedy gives the correct answer.)

(c) (10 points) Look on the web and/or use AI to determine what is known and what is conjectured about these problems.

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2. (30 points) In this problem I will give a property of a domain $\mathbb{D}$ and I want you to give me both:

   - That property expressed as quantifiers.
   - Either a domain $\mathbb{D}$, $\mathbb{D} \subseteq \mathbb{R}$, that has that property or the statement (without proof) that there is no such domain.

   I give an EXAMPLE:

   $\mathbb{D}$ has a least element

   Expressed as quantifiers: $(\exists x \in \mathbb{D})(\forall y \in \mathbb{D})[x \le y]$.

   Domain that works: $\mathbb{N}$ or you could write $\{0, 1, 2, \ldots\}$.

   (a) (10 points) $\mathbb{D}$ is infinite and has both a least element and a greatest element.

   (b) (10 points) Every element $x \in \mathbb{D}$ has both a successor element y (so $x < y$ and there is nothing inbetween) and also a predecessor element w (so $w < x$ and there is nothing inbetween).

   (c) (10 points) $\mathbb{D}$ satisfies all the following properties

      i. $\mathbb{D}$ has a least element.
      ii. $\mathbb{D}$ has a greatest element.
      iii. Every element in $\mathbb{D}$ except the least has a predecessor.
      iv. Every element in $\mathbb{D}$ except the greatest has a successor.

      (So the question here is to determine if such a $\mathbb{D}$ exists, and either give me the $\mathbb{D}$ or state, without proof, there is no $\mathbb{D}$.)

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3. (30 points) In this problem the only symbols you can use are

- The usual logical ones: $\forall$, $\exists$ $\wedge$, $\vee$, $\neg$.
- Variables that range over the domain: $x_1, x_2, x_3, \ldots$.
- These math symbols: $<$, $\le$, $>$, $\ge$, $=$, $\ne$.

For each of the following either give me the sentence I want OR state (without proof) that there is no such sentence.

(a) (15 points) A sentence which is true with domain $\mathbb{Z}$ but false with domain $\mathbb{Q}$.

(b) (15 points) ($(0, 1)$ is the set of all reals between 0 and 1 but not including 0 or 1.) A sentence which is true with domain $(0, 1)$ but false with domain $\mathbb{Q}$. (Warning: You can't use $(\exists x)[x^2 = \frac{1}{2}]$ since this cannot be stated just using $<$.)

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4. **Honors HW 2** This is graded separately as Honors HW02.

   Let $\mathbb{I}$ be the set of irrationals. As usual $\mathbb{Q}$ is the set of rationals.

   (a) (0 points, nothing to hand in) Find someone in the class to play the DUP-SPOILER game $(\mathbb{I}, \mathbb{Q}, 5)$.

   (b) (100 points) Determine which of the following statements is true and give an informal proof of it:

   - There is a k such that SPOILER wins $(\mathbb{I}, \mathbb{Q}, k)$. The informal proof is a value $k \in \mathbb{N}$ and a strategy for SPOILER to win $(\mathbb{I}, \mathbb{Q}, k)$. You DO NOT have to prove that the strategy works, which is why I called it an *informal proof*. BUT the strategy has to be well enough written so that I could carry it out if asked to play the game.
   - There is no k such that SPOILER wins $(\mathbb{I}, \mathbb{Q}, k)$. The informal proof is to give, for all $k \in \mathbb{N}$, a strategy for DUP to win $(\mathbb{I}, \mathbb{Q}, k)$. You DO NOT have to prove that the strategy works, which is why I called it an *informal proof*. BUT the strategy has to be well enough written so that I could carry it out if asked to play the game.