Homework 03, MORALLY Due Feb 24

1. (25 points) For this problem FIRST write the program and make conjectures THEN look up whats true.

   (a) (0 points) Write a program that does the following: For every $1 \leq n \leq 5000$:

   - Using the Dynamic Prog Method (Leo will tell you about that in recitation) to determine natural numbers $(x_1, x_2, \ldots, x_k)$ such that $x_1^4 + \cdots + x_k^4 = n$, with k as small as possible.
   - Below I have the some rows of the output (I didn't use the first x rows since they are boring).

n	k	x_i's
10	10	$10 = 10 \times 1^4$
11	11	$11 = 11 \times 1^4$
12	12	$12 = 12 \times 1^4$
13	13	$13 = 13 \times 1^4$
14	14	$14 = 14 \times 1^4$
15	15	$15 = 14 \times 1^4$
16	1	$16 = 1 \times 2^4$
17	2	$17 = 1 \times 2^4 + 1 \times 1^4$
18	3	$18 = 1 \times 2^4 + 2 \times 1^4$
19	4	$19 = 1 \times 2^4 + 3 \times 1^4$
20	5	$20 = 1 \times 2^4 + 4 \times 1^4$
21	6	$21 = 1 \times 2^4 + 5 \times 1^4$
22	7	$22 = 1 \times 2^4 + 6 \times 1^4$

   DO NOT hand in the program or the output.

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(b) (10 points) Create a table that shows how many numbers ≤ 5000 require 1 fourth-power, 2 fourth-powers, 3 powers, and so on. Below is a sample table (THIS DOES NOT CONTAIN THE CORRECT ANSWER):

# of fourth powers		
1	3	8
2	4	2
3	28	17
$\vdots$	$\vdots$	$\vdots$

(c) (10 points) Based on this data make conjectures of the following forms:

i. Every n is the sum of $\leq XXX$ fourth powers. Write your conjecture in quantifiers.

ii. All but a finite number of n is the sum of $\leq XXX$ fourth powers. Write your conjecture in quantifiers.

(d) (5 points) Look on the web and/or use AI to determine what is known and what is conjectured about these problems.

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2. (25 points)

   (a) (10 points) View the input x, y, z as the number in binary xyz which we denote (xyz). For example, 100 is 4.

   Write a Truth Table for the following function with 3 inputs x, y, z and 1 outputs b.

$$f(x,y,z) = \begin{cases} 0 & \text{if } (xyz) \text{ is NOT PRIME.} \\ 1 & \text{if } (xyz) \text{ is PRIME.} \end{cases}$$

   (NOTE: 0 and 1 are NOT primes. We will discuss this more carefully later.)

   (b) (15 points) Convert your truth table into formulas and give it to us. DO NOT SIMPLIFY.

   (c) (0 points- DO NOT HAND IN) Draw a circuit that computes that truth table.

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3. (25 points) (In this problem we will guide you through a proof that $N(\alpha\beta) = N(\alpha)N(\beta)$.)

   Let $d \in \mathbb{Z}$ and $d \geq 2$. Let $\mathbb{D}_d = \{a + b\sqrt{d} \colon a, b \in \mathbb{Z}\}$.

   Let $\alpha \in \mathbb{D}_d$, $\alpha = a + b\sqrt{d}$. Let $\overline{\alpha} = a - b\sqrt{d}$.

   Let $N(\alpha) = \alpha\overline{\alpha}$. By calculation $N(\alpha) = (a+b\sqrt{d})(a-b\sqrt{d}) = a^2 - db^2$.

   For the questions below let $\alpha = a + b\sqrt{d}$ and $\beta = e + f\sqrt{d}$.

   (a) (0 points-Do Not Hand In Anything) What is $\overline{\alpha\beta}$ in terms of a, b, d, e, f.

   (b) (0 points-Do Not Hand In Anything) What is $\overline{\alpha}\overline{\beta}$ in terms of a, b, d, e, f.

   (c) (0 points-Do Not Hand In Anything) If you did part a a and b correctly then you showed that

   $$\overline{\alpha\beta} = \overline{\alpha}\overline{\beta}.$$

   (d) (25 points) Show that $N(\alpha\beta) = N(\alpha)N(\beta)$
   Hint1: Use $N(x) = x\overline{x}$.
   Hint2: Use $\overline{\alpha\beta} = \overline{\alpha}\overline{\beta}$.
   Hint3: Do not use a, b, d, e, f.

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4. (25 points) Let $\mathbb{D}_d = \{a + b\sqrt{d} \colon a, b \in \mathbb{Z}\}$

   (a) Give an infinite number of units of $\mathbb{D}_3$.

   (b) Give an infinite number of units of $\mathbb{D}_5$.

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5. (0 point-Extra Credit- Graded separately).

   Consider the following problem:

   COUNTSAT: Given a formula ϕ determine how many satisfying assignments it has.

   Clearly if COUNTSAT can be solved quickly then SAT can be solved quickly.

   How about the converse?

   Is the following true: If SAT can be solved quickly then COUNTSAT can be solved quickly.

   LOOK UP what is known about this and give me a WELL WRITTEN writeup of what is known including formal definitions and statement of theorems.