CMSC 250H - HW8, Equivalence Relations

April 20, 2026

Homework 8

Question 2

We have 3 possible sandwich options, 4 fruit options, and 5 dessert options.
a) How many meals can we make in total?
b) How many meals can we make if we can't have apple and applesauce or apple pie in the same meal?

Solution

a) There are $3 \cdot 4 \cdot 5$ *total meals.*
b) There are $3 \cdot 1 \cdot 2$ *invalid meals. So there are* $3 \cdot 4 \cdot 5 - 3 \cdot 1 \cdot 2$ *meals.*

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Question 3

We have 3 possible sandwich options, 4 fruit options, and 5 dessert options. A meal consists of one sandwich, two fruits and 3 desserts.
a) How many meals can we make in total?
b) How many meals can we make if we can't have apple and applesauce or apple pie in the same meal?

Solution

a) There are $\binom{3}{1} \cdot \binom{4}{2} \cdot \binom{5}{3}$ *total meals.*
b) There are $3 \cdot 3 \cdot (\binom{5}{3} - 1)$ *invalid meals. So there are*
$\binom{3}{1} \cdot \binom{4}{2} \cdot \binom{5}{3} - 3 \cdot 3 \cdot (\binom{5}{3} - 1)$

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Question 4

There are 20 students in a room. 10 are left handed, 10 are right handed.

a) How many ways can we form a committee of 8 students so that 4 are LH and 4 are RH?

b) Bob and Abisola do not like each other. How many ways can we form a committee of 8 students so that Bob and Abisola are not in the committee together?

c) Bob and Abisola do not like each other. Bob is LH and Abisola is RH. How many ways can we form a commitee so that 4 are LH and 4 are RH, and Bob and Abisola are not in the committee together.

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Solution

a) There are 10 RH and 10 LH students. Choose 4 from the LH and 4 from the RH

$$\binom{10}{4} \cdot \binom{10}{4}$$

b) Fix Bob and Abisola in the commitee. There are $\binom{18}{6}$ *groups for the last 6. These are invalid There are* $\binom{20}{8}$ *total committees. So the valid committees:*

$$\binom{20}{8} - \binom{18}{6}$$

c) Fix Bob and Abisola in the committee. There are $\binom{9}{3}$ *LH people and* $\binom{9}{3}$ *RH people left to fill the committee, so there are* $\binom{9}{3} \cdot \binom{9}{3}$ *invalid committees. So there are*

$$\binom{10}{4} \cdot \binom{10}{4} - \binom{9}{3} \cdot \binom{9}{3}$$

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Question 5

There are 100 students in a room. 20 are in Kindergarten, 30 are in High School, and 50 are in College.

a) Alice, Bob, and Carol can't be on the same committee. How many ways can we form a committee of 80 people so that all 3 are not in the committee.

b) How many ways can a committee of 80 students be formed where we have 17 kindergartners, 23 high school students, and 40 are college students?

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Solution

a) There are $\binom{100}{80}$ *total committees. Fix Alice, Bob, and Carol on the committee. Then there are* $\binom{97}{77}$ *ways to pick the remaining committee members. So we have*

$$\binom{100}{80} - \binom{97}{77}$$

valid committees.

b) We have $\binom{20}{17}$ *ways to pick kindergartners,* $\binom{30}{23}$ *ways to pick high schoolers, and* $\binom{50}{40}$ *ways to pick college students so:*

$$\binom{20}{17} \cdot \binom{30}{23} \cdot \binom{50}{40}$$

Relations

In math, there is an endless number of ways two entities can be related to each other

1. $6 \in \mathbb{Z}$
2. $6 < 10$
3. $X \subseteq Y$
4. $5 \equiv 1 \mod 2$
5. $3 \nmid 8$

We have a symbol in between two mathematical objects. This symbol is called a relation.

Relations

Definition

A relation $R \subseteq A \times A$. We often say $(x, y) \in R$ or use the notation xRy.

The elements of A are paired together in R if $x \sim y$ (x is related to y). An example: for $A = \{1, 2, 3, 4\}$ consider

$$R = \{(1, 2), (1, 3), (1, 4), (1, 5), (2, 3), (2, 4), (2, 5), (3, 4), (3, 5), (4, 5)\}$$

What mathematical relation does this set represent?

Properties of Relations

1. A relation is **reflexive** if xRx (or equivalently $(x, x) \in R$) for all $a \in A$
2. A relation is **symmetric** if xRy implies yRx for al $x, y \in A$
3. A relation is **transitive** if when xRy and yRz, then xRz for all $x, y, z \in A$

Equivalence Relations and Equivalence Classes

Definition

A relation R is an equivalence relation if it is reflexive, symmetric, and transitive.

Definition

Suppose R is an equivalence relation on a set A. Given an element $a \in A$, the equivalence class containing a is the subset

$$\{x \in A : xRa\}$$

of A consisting of all elements of A that relate to a. We sometimes denote it $[a]$

Example

Is $\geq$ an equivalence relation on $\mathbb{Z}$?

Example

Is $\geq$ an equivalence relation on $\mathbb{Z}$?

Answer: No, it is not symmetric since if $x \geq y$, it could be that $y \not\geq x$

Example

How about mod 5 on $\mathbb{Z}$?

Answer: Yes! The modular operation itself is an equivalence relation! We call the equivalence classes of modular arithmetic, **residue classes**, since they represent the remainder for integer division

Representatives

Even though there may be many elements in an equivalence class, we give the class a representative element.

For example, for mod 5 we denote the equivalence classes

$$[0], [1], [2], [3], [4]$$

or even just

$$0, 1, 2, 3, 4$$

Here, the classes represent the remainders. For example, $[16] = [1]$ and $[127] = [2] \bmod 5$

Example – Infinite Strings of Red and Blue

Consider the set of infinite strings where each symbol is R or B, denoting red or blue respectively. Let's call this set S. For example,

$$\ldots RBRBBBRBBRRR \ldots$$

would be such a string.

Consider the relation

$$R = xRy \qquad \text{if } x \text{ and } y \text{ differ in only a finite amount of spots}$$

Is R an equivalence relation?

Example – Infinite Strings of Red and Blue

1. Reflexive: A string doesn't differ from itself, so it differs in 0 spots – which is finite.
2. Symmetric: If x differs in a finite amount of spots from y, then y clearly also differs from x in a finite amount of spots.
3. Transitive: Suppose x differs from y in m spots and y differs from z in n spots. Then x can only differ from z in at most $m + n$ spots – which is also finite.

So its an equivalence relation! You'll see something like this soon.

Equivalence Classes

Theorem

Let R be an equivalence relation on the set A. Then, for elements $a, b \in A$, $[a] = [b]$ if and only if aRb.

Try proving it yourself! Remember this is a **bi-implication** so we need to prove both directions of the statement.

Equivalence Classes

Proof.

i) $[a] = [b] \implies aRb$

Since we know that $a \in [a]$ and $b \in [b]$, since $[a] = [b]$, then $a \in [b]$ and $b \in [b]$. So aRb, and bRa.

ii) $aRb \implies [a] = [b]$

Suppose $c \in [a]$. Then we know that cRa. Since aRb and R is an equivalence relation, by transitivity cRb. But, then that means every element $c \in [a]$ is contained in $[b]$. Hence, $[a] \subseteq [b]$. The same argument can be used to show $[b] \subseteq [a]$, establishing $[a] = [b]$ ☐

Partition

Definition

A partition of a set A is a set of subsets $\{A_i : A_i \subseteq A\}$ of A that union to form A, with each element being in one unique subset. That is

$$\bigcup_i A_i = A \quad \text{with} \quad A_i \cap A_j = \emptyset$$

for all $i < j$

Example: For $A = \{1, 2, 3, 4, 5\}$

$$A_1 = \{1\}, A_2 = \{4, 5\}, A_3 = \{2, 3\}$$

is partition of A of size 3.

Equivalence Classes and Partitions

Theorem

The equivalence classes of an equivalence relation, R, on a set A, form a partition of A.

Proof.

We need to show two things:

1. $A = \bigcup_{a \in A} [a]$
2. For any $a \neq b$, $[a] \cap [b] = \emptyset$

☐

Equivalence Classes and Partitions – 1

Proof.

Consider $x \in A$. Then, since R is an equivalence relation and reflexive, $x \in [x]$. Thus,

$$x \in \bigcup_{a \in A} [a]$$

for some $a \in A$ and

$$A \subseteq \bigcup_{a \in A} [a]$$

Now, suppose $x \in \bigcup_{a \in A}[a]$. Then, $x \in [a]$ for some $a \in A$. Since $[a] \subseteq A$, it must be that $x \in A$. So

$$\bigcup_{a \in A} [a] \subseteq A$$

Hence $A = \bigcup_{a \in A}[a]$ □

Equivalence Classes and Partitions – 2

Proof.

We prove 2 using the contrapositive. Suppose that $[a] \bigcap [b] \neq \emptyset$ for $a, b \in A$. Then, there must exist some element $c \in [a] \bigcap [b]$. Thus, $c \in [a]$ and $c \in [b]$. Hence cRa and cRb. By symmetry and transitivity,

$$aRc \text{ and } cRb \implies aRb$$

so by the previous theorem $[a] = [b]$, so $[a] \neq [b]$ is not true, proving the theorem. □