BILL, RECORD LECTURE!!!!

BILL RECORD LECTURE!!!

Small Ramsey Numbers

Exposition by William Gasarch

April 23, 2026

Lets Party Like Its January of 2019

Recall the first theorem one usually hears in Ramsey Theory and can tell your non-math friends about.

> *If there are 6 people at a party, either 3 know each other or 3 do not know each other.*

Lets Party Like Its January of 2019

Recall the first theorem one usually hears in Ramsey Theory and can tell your non-math friends about.

> *If there are 6 people at a party, either 3 know each other or 3 do not know each other.*

We define graphs and complete graphs and state this theorem in those terms.

Graphs and Complete Graphs

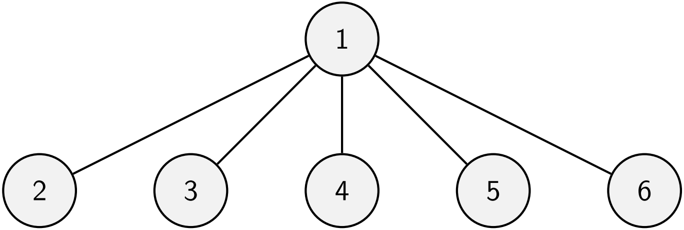
Def A **Graph** $G = (V, E)$ is a set V and a set of unordered pairs from V, called edges. These can easily be drawn.

Graphs and Complete Graphs

Def A **Graph** $G = (V, E)$ is a set V and a set of unordered pairs from V, called edges. These can easily be drawn.
Example

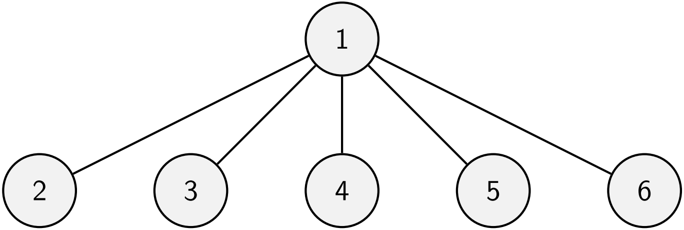
Graphs and Complete Graphs

Def A **Graph** $G = (V, E)$ is a set V and a set of unordered pairs from V, called edges. These can easily be drawn.
Example



Graphs and Complete Graphs

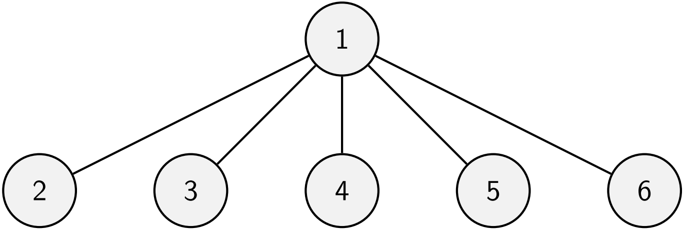
Def A **Graph** $G = (V, E)$ is a set V and a set of unordered pairs from V, called edges. These can easily be drawn.
Example



$V = \{1, 2, 3, 4, 5, 6\}$.

Graphs and Complete Graphs

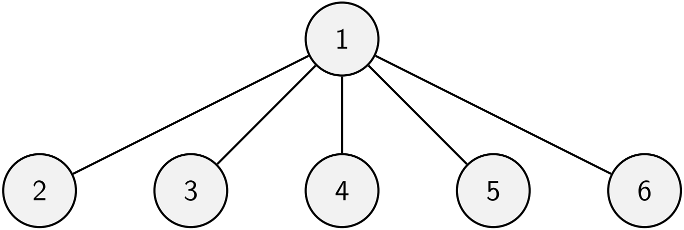
Def A **Graph** $G = (V, E)$ is a set V and a set of unordered pairs from V, called edges. These can easily be drawn.
Example



$V = \{1, 2, 3, 4, 5, 6\}$.
$E = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 5\}, \{1, 6\}\}$.

Graphs and Complete Graphs

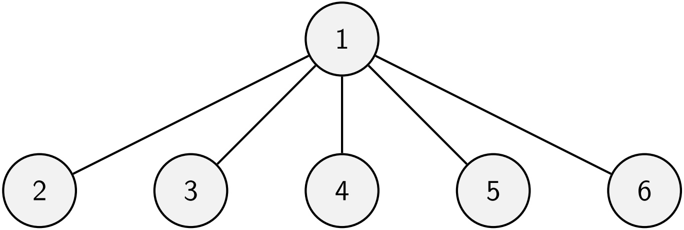
Def A **Graph** $G = (V, E)$ is a set V and a set of unordered pairs from V, called edges. These can easily be drawn.
Example



$V = \{1, 2, 3, 4, 5, 6\}$.
$E = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 5\}, \{1, 6\}\}$.
Def The **degree (deg)** of a vertex is how many edges use it.

Graphs and Complete Graphs

Def A **Graph** $G = (V, E)$ is a set V and a set of unordered pairs from V, called edges. These can easily be drawn.
Example



$V = \{1, 2, 3, 4, 5, 6\}$.
$E = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 5\}, \{1, 6\}\}$.
Def The **degree (deg)** of a vertex is how many edges use it.
In the above graph $\deg(1) = 5$ and
$\deg(2) = \deg(3) = \deg(4) = \deg(5) = \deg(6) = 1$.

Complete Graphs

Def The **Complete Graph on n Vertices**, denoted K_n, is $V = \{1, \ldots, n\}$ and E is **all** possible edges.

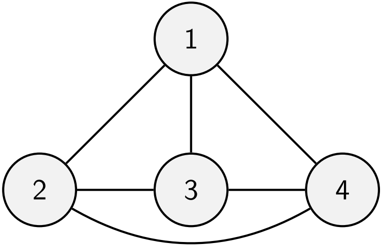
Complete Graphs

Def The **Complete Graph on n Vertices**, denoted K_n, is $V = \{1, \ldots, n\}$ and E is **all** possible edges.
Example

Complete Graphs

Def The **Complete Graph on n Vertices**, denoted K_n, is $V = \{1, \ldots, n\}$ and E is **all** possible edges.

Example

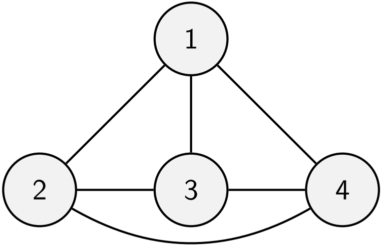


This graph is K_4.

Complete Graphs

Def The **Complete Graph on n Vertices**, denoted K_n, is $V = \{1, \ldots, n\}$ and E is **all** possible edges.

Example



This graph is K_4.

Note Every vertex of K_n has degree $n - 1$.

More Notation

Below is standard notation which you may or may not have seen.

More Notation

Below is standard notation which you may or may not have seen.

Thats a tautology!

More Notation

Below is standard notation which you may or may not have seen.

Thats a tautology!

Notation

More Notation

Below is standard notation which you may or may not have seen.

Thats a tautology!

Notation

- $\exists$ means **there exists**

More Notation

Below is standard notation which you may or may not have seen.

Thats a tautology!

Notation

- $\exists$ means **there exists**
- $\forall$ means **for all**

One More Definition

Def Let $G = (V, E)$ be a graph. Let $U \subseteq V$.

One More Definition

Def Let $G = (V, E)$ be a graph. Let $U \subseteq V$.

1. U is a **Clique** if all of the verts in U have an edge between them.

One More Definition

Def Let $G = (V, E)$ be a graph. Let $U \subseteq V$.

1. U is a **Clique** if all of the verts in U have an edge between them.
2. If $|U| = k$ then we may call U a k-**clique**.

One More Definition

Def Let $G = (V, E)$ be a graph. Let $U \subseteq V$.

1. U is a **Clique** if all of the verts in U have an edge between them.
2. If $|U| = k$ then we may call U a k-**clique**.
3. If the edges of G are 2-colored with **RED** and **BLUE**, and all of the edges between verts of U are **RED** then we call U a **Red Clique**. Similar for **Blue**.

One More Definition

Def Let $G = (V, E)$ be a graph. Let $U \subseteq V$.

1. U is a **Clique** if all of the verts in U have an edge between them.
2. If $|U| = k$ then we may call U a k-**clique**.
3. If the edges of G are 2-colored with **RED** and **BLUE**, and all of the edges between verts of U are **RED** then we call U a **Red Clique**. Similar for **Blue**.
4. If I formed a rock band it would be called

One More Definition

Def Let $G = (V, E)$ be a graph. Let $U \subseteq V$.

1. U is a **Clique** if all of the verts in U have an edge between them.
2. If $|U| = k$ then we may call U a k-**clique**.
3. If the edges of G are 2-colored with **RED** and **BLUE**, and all of the edges between verts of U are **RED** then we call U a **Red Clique**. Similar for **Blue**.
4. If I formed a rock band it would be called

   Bill Gasarch and the Red Cliques!

The First Theorem, Restated

For every 2-coloring of the edges of K_6 *there is a monochromatic* K_3 *(triangle).*

The First Theorem, Restated

> *For every 2-coloring of the edges of K_6 there is a monochromatic K_3 (triangle).*

From now on we use **mono** for **monochromatic**

The First Theorem, Restated

For every 2-coloring of the edges of K_6 there is a monochromatic K_3 (triangle).

From now on we use **mono** for **monochromatic**
We could state that as

$\forall$ *2-coloring of the edges of* K_6 $\exists$ *a mono* K_3 *(triangle).*

The First Theorem, Restated

For every 2-coloring of the edges of K_6 there is a monochromatic K_3 (triangle).

From now on we use **mono** for **monochromatic**

We could state that as

$\forall$ *2-coloring of the edges of* K_6 $\exists$ *a mono* K_3 *(triangle).*

We could state that as

$\forall$ *2-coloring of the edges of* K_6 $\exists$ *a mono 3-clique (triangle).*

The First Theorem, Restated

For every 2-coloring of the edges of K_6 there is a monochromatic K_3 (triangle).

From now on we use **mono** for **monochromatic**

We could state that as

$\forall$ *2-coloring of the edges of* K_6 $\exists$ *a mono* K_3 *(triangle).*

We could state that as

$\forall$ *2-coloring of the edges of* K_6 $\exists$ *a mono 3-clique (triangle).*

We prove this in the next few slides.

The First Theorem in Ramsey Theory

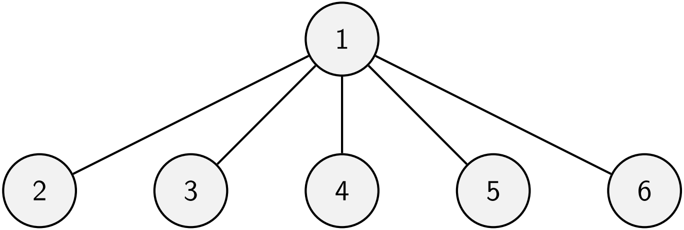
Thm For all 2-colorings of the edges of K_6 there is a mono K_3.

Focus on Vertex 1

Given a 2-coloring of the edges of K_6 we look at vertex 1.

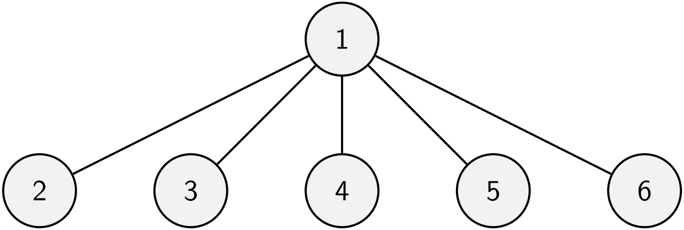
Focus on Vertex 1

Given a 2-coloring of the edges of K_6 we look at vertex 1.



Focus on Vertex 1

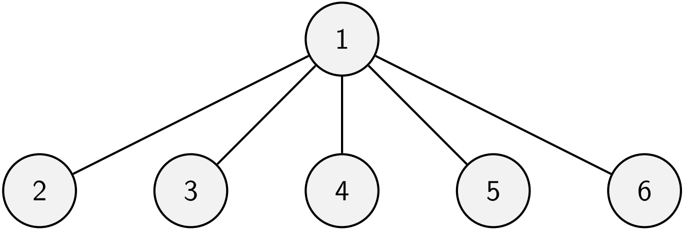
Given a 2-coloring of the edges of K_6 we look at vertex 1.



There are 5 edges coming out of vertex 1.

Focus on Vertex 1

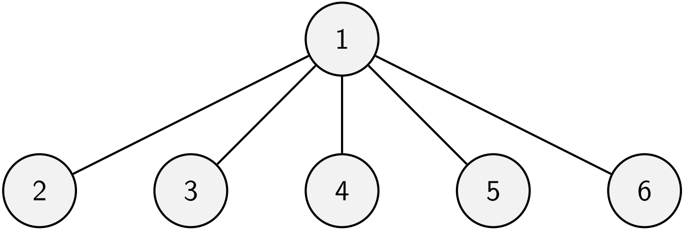
Given a 2-coloring of the edges of K_6 we look at vertex 1.



There are 5 edges coming out of vertex 1.
They are 2 colored.

Focus on Vertex 1

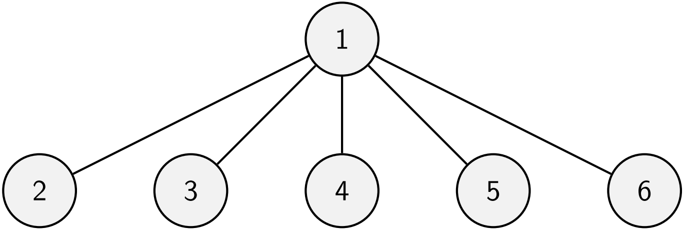
Given a 2-coloring of the edges of K_6 we look at vertex 1.



There are 5 edges coming out of vertex 1.
They are 2 colored.
$\exists$ 3 edges from vertex 1 that are the same color.

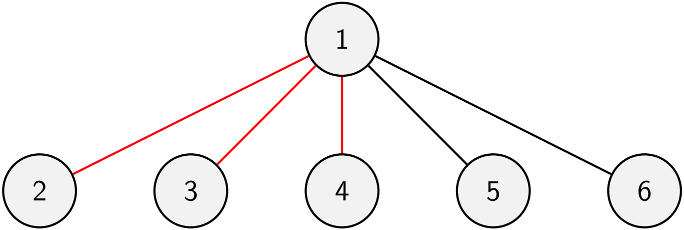
Focus on Vertex 1

Given a 2-coloring of the edges of K_6 we look at vertex 1.

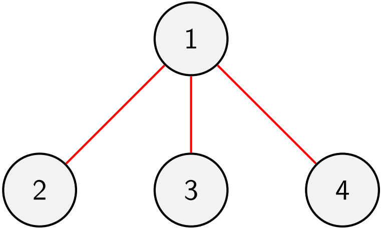


There are 5 edges coming out of vertex 1.
They are 2 colored.
$\exists$ 3 edges from vertex 1 that are the same color.
We can assume $(1,2)$, $(1,3)$, $(1,4)$ are all **RED**.

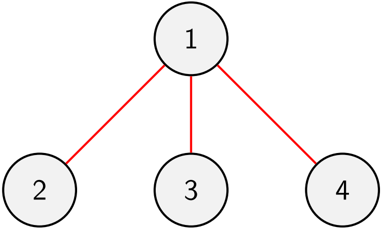
(1,2), (1,3), (1,4) are RED



We Look Just at Vertices 1,2,3,4



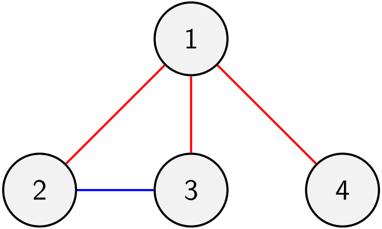
We Look Just at Vertices 1,2,3,4



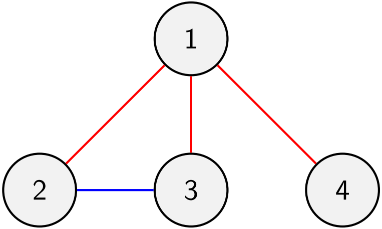
If $(2,3)$ is **RED** then get **RED** Triangle. So assume (2,3) is **BLUE**.

(2,3) is BLUE

(2,3) is BLUE



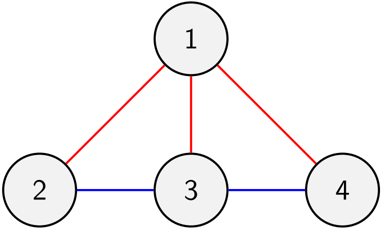
(2,3) is BLUE



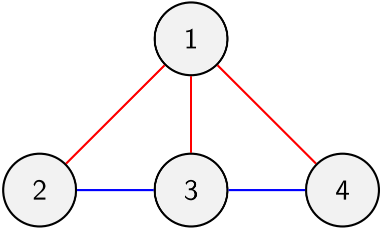
If (3,4) is **RED** then get **RED** triangle. So assume (3,4) is **BLUE**.

(2,3) and (3,4) are BLUE

(2,3) and (3,4) are BLUE



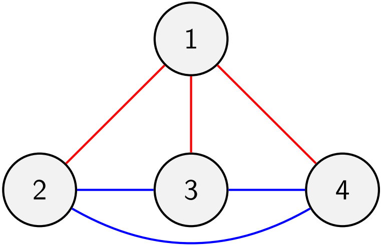
(2,3) and (3,4) are **BLUE**



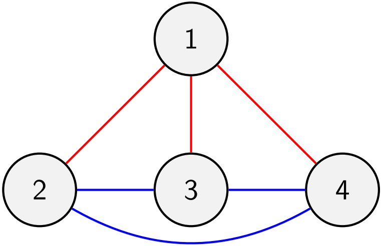
If (2,4) is **RED** then get **RED** triangle. So assume (2,4) is **BLUE**.

(2,4) is BLUE

(2,4) is BLUE



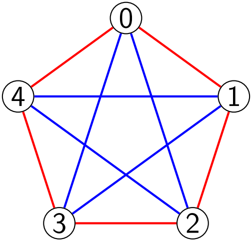
(2,4) is **BLUE**



Note that there is a **BLUE** triangle with verts $2, 3, 4$. Done!

What If We Color Edges Of K_5?

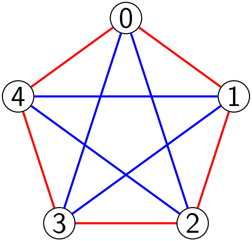
What If We Color Edges Of K_5?



This graph is not arbitrary.
$\mathrm{SQ}_5 = \{x^2 \pmod 5 : 0 \le x \le 4\} = \{0, 1, 4\}$.

- If $i - j \in \mathrm{SQ}_5$ then **RED**.
- If $i - j \notin \mathrm{SQ}_5$ then **BLUE**.

What If We Color Edges Of K_5?



This graph is not arbitrary.
$\mathrm{SQ}_5 = \{x^2 \pmod 5 : 0 \le x \le 4\} = \{0, 1, 4\}$.

- If $i - j \in \mathrm{SQ}_5$ then **RED**.
- If $i - j \notin \mathrm{SQ}_5$ then **BLUE**.

Two ways to show no mono $\triangle$s on next slide.

No Mono $\triangle$s

No Mono $\triangle$s

Need to show there are no mono $\triangle$.

No Mono $\triangle$s

Need to show there are no mono $\triangle$.

Method 1

No Mono $\triangle$s

Need to show there are no mono $\triangle$.

Method 1

The **R** edges are all symmetric so if there is a $\triangle$ edge can assume one of the edges is $(0, 1)$. No x with $\mathrm{COL}(0, x) = \mathrm{COL}(0, 1) =$ **R**.

No Mono $\triangle$s

Need to show there are no mono $\triangle$.

Method 1

The **R** edges are all symmetric so if there is a $\triangle$ edge can assume one of the edges is $(0, 1)$. No x with $\mathrm{COL}(0, x) = \mathrm{COL}(0, 1) =$ **R**.

The **B** edges are all symmetric so if there is a $\triangle$ can assume one of the edges is $(0, 2)$. No x with $\mathrm{COL}(0, x) = \mathrm{COL}(0, 1) =$ **B**.

No Mono $\triangle$s

Need to show there are no mono $\triangle$.

Method 1

The **R** edges are all symmetric so if there is a $\triangle$ edge can assume one of the edges is $(0,1)$. No x with $\mathrm{COL}(0,x) = \mathrm{COL}(0,1) =$ **R**.

The **B** edges are all symmetric so if there is a $\triangle$ can assume one of the edges is $(0,2)$. No x with $\mathrm{COL}(0,x) = \mathrm{COL}(0,1) =$ **B**.

PRO Easy Proof.

No Mono $\triangle$s

Need to show there are no mono $\triangle$.

Method 1

The **R** edges are all symmetric so if there is a $\triangle$ edge can assume one of the edges is $(0,1)$. No x with $\mathrm{COL}(0,x) = \mathrm{COL}(0,1) =$ **R**.
The **B** edges are all symmetric so if there is a $\triangle$ can assume one of the edges is $(0,2)$. No x with $\mathrm{COL}(0,x) = \mathrm{COL}(0,1) =$ **B**.
PRO Easy Proof. **CON** didn't use SQ_5 so unlikely to generalize.

No Mono $\triangle$s

Need to show there are no mono $\triangle$.

Method 1

The **R** edges are all symmetric so if there is a $\triangle$ edge can assume one of the edges is $(0, 1)$. No x with $\mathrm{COL}(0, x) = \mathrm{COL}(0, 1) =$ **R**.
The **B** edges are all symmetric so if there is a $\triangle$ can assume one of the edges is $(0, 2)$. No x with $\mathrm{COL}(0, x) = \mathrm{COL}(0, 1) =$ **B**.
PRO Easy Proof. **CON** didn't use SQ_5 so unlikely to generalize.

Method 2 All $\equiv$ are mod 5.

No Mono $\triangle$s

Need to show there are no mono $\triangle$.

Method 1

The **R** edges are all symmetric so if there is a $\triangle$ edge can assume one of the edges is $(0, 1)$. No x with $\mathrm{COL}(0, x) = \mathrm{COL}(0, 1) =$ **R**.

The **B** edges are all symmetric so if there is a $\triangle$ can assume one of the edges is $(0, 2)$. No x with $\mathrm{COL}(0, x) = \mathrm{COL}(0, 1) =$ **B**.

PRO Easy Proof. **CON** didn't use SQ_5 so unlikely to generalize.

Method 2 All $\equiv$ are mod 5.

1) Assume a, b, c form a $\triangle$. Then $a - b, b - c, c - a \in \mathrm{SQ}_5$.

$a - b \equiv x^2$, $b - c \equiv y^2$, $c - a \equiv z^2$.

No Mono $\triangle$s

Need to show there are no mono $\triangle$.

Method 1

The **R** edges are all symmetric so if there is a $\triangle$ edge can assume one of the edges is $(0,1)$. No x with $\mathrm{COL}(0,x) = \mathrm{COL}(0,1) =$ **R**.
The **B** edges are all symmetric so if there is a $\triangle$ can assume one of the edges is $(0,2)$. No x with $\mathrm{COL}(0,x) = \mathrm{COL}(0,1) =$ **B**.
PRO Easy Proof. **CON** didn't use SQ_5 so unlikely to generalize.

Method 2 All $\equiv$ are mod 5.
1) Assume a, b, c form a $\triangle$. Then $a-b, b-c, c-a \in \mathrm{SQ}_5$.
$a - b \equiv x^2$, $b - c \equiv y^2$, $c - a \equiv z^2$.
$x^2 + y^2 + z^2 \equiv 0$.

No Mono $\triangle$s

Need to show there are no mono $\triangle$.

Method 1

The **R** edges are all symmetric so if there is a $\triangle$ edge can assume one of the edges is $(0,1)$. No x with $\mathrm{COL}(0,x) = \mathrm{COL}(0,1) = $ **R**.
The **B** edges are all symmetric so if there is a $\triangle$ can assume one of the edges is $(0,2)$. No x with $\mathrm{COL}(0,x) = \mathrm{COL}(0,1) = $ **B**.
PRO Easy Proof. **CON** didn't use SQ_5 so unlikely to generalize.

Method 2 All $\equiv$ are mod 5.
1) Assume a, b, c form a $\triangle$. Then $a-b, b-c, c-a \in \mathrm{SQ}_5$.
$a - b \equiv x^2$, $b - c \equiv y^2$, $c - a \equiv z^2$.
$x^2 + y^2 + z^2 \equiv 0$. Can show this implies $x = y = z = 0$.

No Mono $\triangle$s

Need to show there are no mono $\triangle$.

Method 1

The **R** edges are all symmetric so if there is a $\triangle$ edge can assume one of the edges is $(0,1)$. No x with $\mathrm{COL}(0,x) = \mathrm{COL}(0,1) =$ **R**.

The **B** edges are all symmetric so if there is a $\triangle$ can assume one of the edges is $(0,2)$. No x with $\mathrm{COL}(0,x) = \mathrm{COL}(0,1) =$ **B**.

PRO Easy Proof. **CON** didn't use SQ_5 so unlikely to generalize.

Method 2 All $\equiv$ are mod 5.

1) Assume a,b,c form a $\triangle$. Then $a-b, b-c, c-a \in \mathrm{SQ}_5$.
$a-b \equiv x^2$, $b-c \equiv y^2$, $c-a \equiv z^2$.
$x^2+y^2+z^2 \equiv 0$. Can show this implies $x=y=z=0$.

2) Assume a,b,c form a $\triangle$. Then $a-b, b-c, c-a \notin \mathrm{SQ}_5$. For all p, product of 2 nonsquares mod p is in SQ_p (HW). $2 \notin \mathrm{SQ}_5$ so $2(a-b)=x^2$, $2(b-c)=y^2$, $2(c-a)=z^2$.

No Mono $\triangle$s

Need to show there are no mono $\triangle$.

Method 1

The **R** edges are all symmetric so if there is a $\triangle$ edge can assume one of the edges is $(0,1)$. No x with $\mathrm{COL}(0,x) = \mathrm{COL}(0,1) =$ **R**.
The **B** edges are all symmetric so if there is a $\triangle$ can assume one of the edges is $(0,2)$. No x with $\mathrm{COL}(0,x) = \mathrm{COL}(0,1) =$ **B**.
PRO Easy Proof. **CON** didn't use SQ_5 so unlikely to generalize.

Method 2 All $\equiv$ are mod 5.
1) Assume a, b, c form a $\triangle$. Then $a-b, b-c, c-a \in \mathrm{SQ}_5$.
$a-b \equiv x^2$, $b-c \equiv y^2$, $c-a \equiv z^2$.
$x^2+y^2+z^2 \equiv 0$. Can show this implies $x=y=z=0$.
2) Assume a, b, c form a $\triangle$. Then $a-b, b-c, c-a \notin \mathrm{SQ}_5$. For all p, product of 2 nonsquares mod p is in SQ_p (HW). $2 \notin \mathrm{SQ}_5$ so
$2(a-b) = x^2$, $2(b-c) = y^2$, $2(c-a) = z^2$.
$2(x^2+y^2+z^2) \equiv 0$.

No Mono $\triangle$s

Need to show there are no mono $\triangle$.

Method 1

The **R** edges are all symmetric so if there is a $\triangle$ edge can assume one of the edges is $(0,1)$. No x with $\mathrm{COL}(0,x) = \mathrm{COL}(0,1) =$ **R**.
The **B** edges are all symmetric so if there is a $\triangle$ can assume one of the edges is $(0,2)$. No x with $\mathrm{COL}(0,x) = \mathrm{COL}(0,1) =$ **B**.
PRO Easy Proof. **CON** didn't use SQ_5 so unlikely to generalize.

Method 2 All $\equiv$ are mod 5.
1) Assume a, b, c form a $\triangle$. Then $a-b, b-c, c-a \in \mathrm{SQ}_5$.
$a-b \equiv x^2$, $b-c \equiv y^2$, $c-a \equiv z^2$.
$x^2+y^2+z^2 \equiv 0$. Can show this implies $x=y=z=0$.
2) Assume a, b, c form a $\triangle$. Then $a-b, b-c, c-a \notin \mathrm{SQ}_5$. For all p, product of 2 nonsquares mod p is in SQ_p (HW). $2 \notin \mathrm{SQ}_5$ so
$2(a-b) = x^2$, $2(b-c) = y^2$, $2(c-a) = z^2$.
$2(x^2+y^2+z^2) \equiv 0$. Divide by 2 (mult by 3) to get
$x^2+y^2+z^2 \equiv 0$ which implies $x=y=z$.

No Mono △s

Need to show there are no mono △.

Method 1

The **R** edges are all symmetric so if there is a △ edge can assume one of the edges is $(0,1)$. No x with $\mathrm{COL}(0,x) = \mathrm{COL}(0,1) =$ **R**.

The **B** edges are all symmetric so if there is a △ can assume one of the edges is $(0,2)$. No x with $\mathrm{COL}(0,x) = \mathrm{COL}(0,1) =$ **B**.

PRO Easy Proof. **CON** didn't use SQ_5 so unlikely to generalize.

Method 2 All $\equiv$ are mod 5.

1) Assume a, b, c form a △. Then $a-b, b-c, c-a \in \mathrm{SQ}_5$.

$a-b \equiv x^2$, $b-c \equiv y^2$, $c-a \equiv z^2$.

$x^2+y^2+z^2 \equiv 0$. Can show this implies $x=y=z=0$.

2) Assume a, b, c form a △. Then $a-b, b-c, c-a \notin \mathrm{SQ}_5$. For all p, product of 2 nonsquares mod p is in SQ_p (HW). $2 \notin \mathrm{SQ}_5$ so

$2(a-b) = x^2$, $2(b-c) = y^2$, $2(c-a) = z^2$.

$2(x^2+y^2+z^2) \equiv 0$. Divide by 2 (mult by 3) to get

$x^2+y^2+z^2 \equiv 0$ which implies $x=y=z$.

UPSHOT $R(3,3)=6$.

Asymmetric Ramsey Numbers

Definition Let $a, b \geq 2$. $R(a, b)$ is least n such that for all 2-colorings of K_n there is **either** a red K_a or a blue K_b.

1. $R(a, b) = R(b, a)$.
2. $R(2, b) = b$
3. $R(a, 2) = a$

Asymmetric Ramsey Numbers

Definition Let $a, b \geq 2$. $R(a, b)$ is least n such that for all 2-colorings of K_n there is **either** a red K_a or a blue K_b.

1. $R(a, b) = R(b, a)$.
2. $R(2, b) = b$
3. $R(a, 2) = a$

Proof left to the reader, but its easy.

$R(a,b) \leq R(a-1,b) + R(a,b-1)$

Theorem $R(a,b) \leq R(a-1,b) + R(a,b-1)$

$R(a,b) \leq R(a-1,b) + R(a,b-1)$

Theorem $R(a,b) \leq R(a-1,b) + R(a,b-1)$

Let $n = R(a-1,b) + R(a,b-1)$.

$R(a, b) \leq R(a-1, b) + R(a, b-1)$

Theorem $R(a, b) \leq R(a-1, b) + R(a, b-1)$

Let $n = R(a-1, b) + R(a, b-1)$.

Assume you have a coloring of the edges of K_n.

$R(a, b) \leq R(a-1, b) + R(a, b-1)$

Theorem $R(a, b) \leq R(a-1, b) + R(a, b-1)$

Let $n = R(a-1, b) + R(a, b-1)$.
Assume you have a coloring of the edges of K_n.
The proof has three cases on the next three slides.

$R(a,b) \leq R(a-1,b) + R(a,b-1)$

Theorem $R(a,b) \leq R(a-1,b) + R(a,b-1)$

Let $n = R(a-1,b) + R(a,b-1)$.
Assume you have a coloring of the edges of K_n.
The proof has three cases on the next three slides.
They will be

$R(a,b) \le R(a-1,b) + R(a,b-1)$

Theorem $R(a,b) \le R(a-1,b) + R(a,b-1)$

Let $n = R(a-1,b) + R(a,b-1)$.
Assume you have a coloring of the edges of K_n.
The proof has three cases on the next three slides.
They will be

1. There is a vertex with large **Red** Deg.

$R(a, b) \leq R(a-1, b) + R(a, b-1)$

Theorem $R(a, b) \leq R(a-1, b) + R(a, b-1)$
Let $n = R(a-1, b) + R(a, b-1)$.
Assume you have a coloring of the edges of K_n.
The proof has three cases on the next three slides.
They will be

1. There is a vertex with large **Red** Deg.
2. There is a vertex with large **Blue** Deg.

$R(a, b) \leq R(a-1, b) + R(a, b-1)$

Theorem $R(a, b) \leq R(a-1, b) + R(a, b-1)$
Let $n = R(a-1, b) + R(a, b-1)$.
Assume you have a coloring of the edges of K_n.
The proof has three cases on the next three slides.
They will be

1. There is a vertex with large **Red** Deg.
2. There is a vertex with large **Blue** Deg.
3. All verts have small **Red** degree and small **Blue** degree.

Some Vertex v Has Large Red Deg

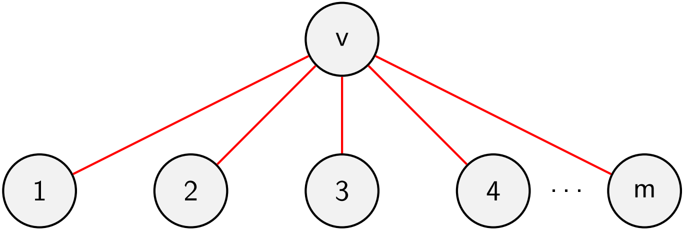
Case 1 $(\exists v)[\deg_R(v) \ge R(a-1, b)]$.

Some Vertex v Has Large Red Deg

Case 1 $(\exists v)[\deg_R(v) \geq R(a-1,b)]$.
Let $m = R(a-1,b)$.

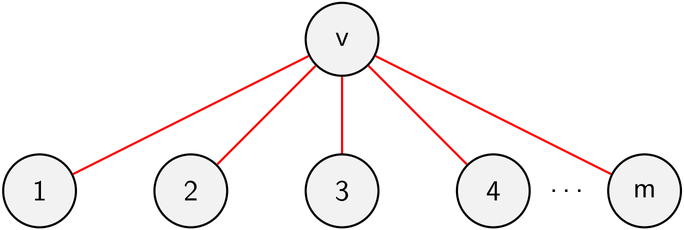
Some Vertex v Has Large Red Deg

Case 1 $(\exists v)[\deg_R(v) \ge R(a-1,b)]$.
Let $m = R(a-1,b)$.



Some Vertex v Has Large **Red** Deg

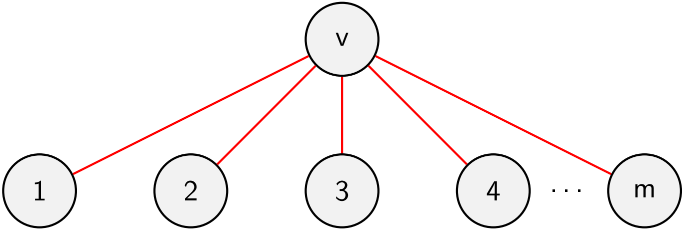
Case 1 $(\exists v)[\deg_R(v) \ge R(a-1,b)]$.
Let $m = R(a-1,b)$.



Case 1.1 There is a **Red** K_{a-1} in $\{1,\ldots,m\}$. This set together with vertex v is a **Red** K_a.

Some Vertex v Has Large Red Deg

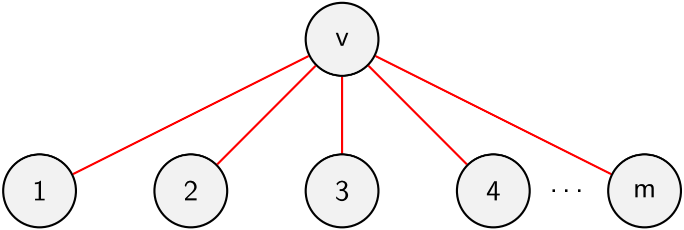
Case 1 $(\exists v)[\deg_R(v) \ge R(a-1,b)]$.
Let $m = R(a-1,b)$.



Case 1.1 There is a **Red** K_{a-1} in $\{1,\ldots,m\}$. This set together with vertex v is a **Red** K_a.
Case 1.2 There is a **Blue** K_b in $\{1,\ldots,m\}$. DONE.

Some Vertex v Has Large Red Deg

Case 1 $(\exists v)[\deg_R(v) \geq R(a-1,b)]$.
Let $m = R(a-1,b)$.



Case 1.1 There is a **Red** K_{a-1} in $\{1,\ldots,m\}$. This set together with vertex v is a **Red** K_a.
Case 1.2 There is a **Blue** K_b in $\{1,\ldots,m\}$. DONE.
Case 1.3 Neither. **Impossible** since $m = R(a-1,b)$.

Some Vertex v Has Large Blue Deg

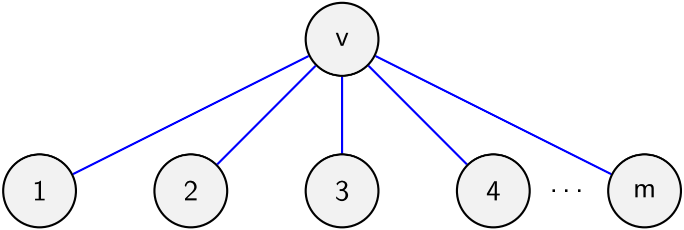
Case 2 $(\exists v)[\deg_B(v) \ge R(a, b-1)]$.

Some Vertex v Has Large Blue Deg

Case 2 $(\exists v)[\deg_B(v) \geq R(a, b-1)]$.
Let $m = R(a, b-1)$.

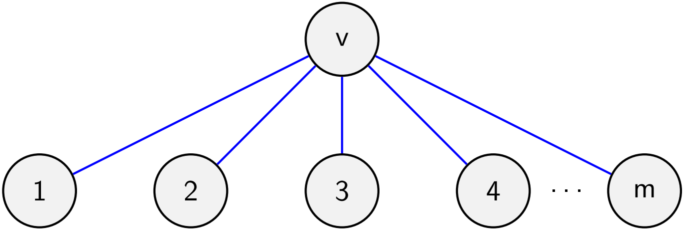
Some Vertex v Has Large Blue Deg

Case 2 $(\exists v)[\deg_B(v) \ge R(a, b-1)]$.
Let $m = R(a, b-1)$.



Some Vertex v Has Large Blue Deg

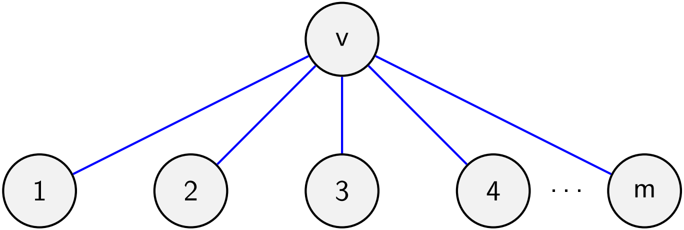
Case 2 $(\exists v)[\deg_B(v) \geq R(a, b-1)]$.
Let $m = R(a, b-1)$.



Case 2.1 There is a **Red** K_a in $\{1, \ldots, m\}$. DONE

Some Vertex v Has Large Blue Deg

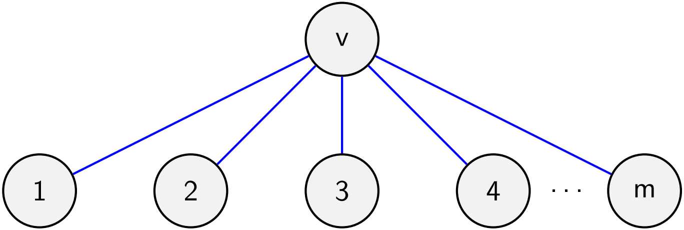
Case 2 $(\exists v)[\deg_B(v) \ge R(a, b-1)]$.
Let $m = R(a, b-1)$.



Case 2.1 There is a **Red** K_a in $\{1, \ldots, m\}$. DONE
Case 2.2 There is a **Blue** K_{b-1} in $\{1, \ldots, m\}$. This set together with vertex v is a **Blue** K_b.

Some Vertex v Has Large Blue Deg

Case 2 $(\exists v)[\deg_B(v) \ge R(a, b-1)]$.
Let $m = R(a, b-1)$.



Case 2.1 There is a **Red** K_a in $\{1, \ldots, m\}$. DONE
Case 2.2 There is a **Blue** K_{b-1} in $\{1, \ldots, m\}$. This set together with vertex v is a **Blue** K_b.
Case 2.3 Neither. **Impossible** since $m = R(a, b-1)$.

All Verts: Small Red Deg and Small Blue Deg

Case 3 Negate Case 1 and Case 2:

All Verts: Small Red Deg and Small Blue Deg

Case 3 Negate Case 1 and Case 2:

1. $(\forall v)[\deg_R(v) \le R(a-1,b)-1]$ and

All Verts: Small Red Deg and Small Blue Deg

Case 3 Negate Case 1 and Case 2:

1. $(\forall v)[\deg_R(v) \le R(a-1,b)-1]$ and
2. $(\forall v)[\deg_B(v) \le R(a,b-1)-1]$

All Verts: Small Red Deg and Small Blue Deg

Case 3 Negate Case 1 and Case 2:

1. $(\forall v)[\deg_R(v) \le R(a-1,b)-1]$ and
2. $(\forall v)[\deg_B(v) \le R(a,b-1)-1]$

Hence

$$(\forall v)[\deg(v) \le R(a-1,b)+R(a,b-1)-2 = n-2]$$

All Verts: Small Red Deg and Small Blue Deg

Case 3 Negate Case 1 and Case 2:

1. $(\forall v)[\deg_R(v) \le R(a-1,b)-1]$ and
2. $(\forall v)[\deg_B(v) \le R(a,b-1)-1]$

Hence

$$(\forall v)[\deg(v) \le R(a-1,b)+R(a,b-1)-2 = n-2]$$

Not possible since every vertex of K_n has degree $n-1$.

Lets Compute Bounds on $R(a, b)$

Lets Compute Bounds on $R(a, b)$

- $R(3,3) \leq R(2,3) + R(3,2) \leq 3 + 3 = 6$

Lets Compute Bounds on $R(a, b)$

- $R(3,3) \leq R(2,3) + R(3,2) \leq 3 + 3 = 6$
- $R(3,4) \leq R(2,4) + R(3,3) \leq 4 + 6 = 10$

Lets Compute Bounds on $R(a, b)$

- $R(3,3) \leq R(2,3) + R(3,2) \leq 3 + 3 = 6$
- $R(3,4) \leq R(2,4) + R(3,3) \leq 4 + 6 = 10$
- $R(3,5) \leq R(2,5) + R(3,4) \leq 5 + 10 = 15$

Lets Compute Bounds on $R(a, b)$

- $R(3,3) \leq R(2,3) + R(3,2) \leq 3 + 3 = 6$
- $R(3,4) \leq R(2,4) + R(3,3) \leq 4 + 6 = 10$
- $R(3,5) \leq R(2,5) + R(3,4) \leq 5 + 10 = 15$
- $R(3,6) \leq R(2,6) + R(3,5) \leq 6 + 15 = 21$

Lets Compute Bounds on $R(a, b)$

- $R(3,3) \leq R(2,3) + R(3,2) \leq 3 + 3 = 6$
- $R(3,4) \leq R(2,4) + R(3,3) \leq 4 + 6 = 10$
- $R(3,5) \leq R(2,5) + R(3,4) \leq 5 + 10 = 15$
- $R(3,6) \leq R(2,6) + R(3,5) \leq 6 + 15 = 21$
- $R(3,7) \leq R(2,7) + R(3,6) \leq 7 + 21 = 28$

Lets Compute Bounds on $R(a, b)$

- $R(3,3) \leq R(2,3) + R(3,2) \leq 3 + 3 = 6$
- $R(3,4) \leq R(2,4) + R(3,3) \leq 4 + 6 = 10$
- $R(3,5) \leq R(2,5) + R(3,4) \leq 5 + 10 = 15$
- $R(3,6) \leq R(2,6) + R(3,5) \leq 6 + 15 = 21$
- $R(3,7) \leq R(2,7) + R(3,6) \leq 7 + 21 = 28$
- $R(4,4) \leq R(3,4) + R(4,3) \leq 10 + 10 = 20$

Lets Compute Bounds on $R(a, b)$

- $R(3,3) \leq R(2,3) + R(3,2) \leq 3 + 3 = 6$
- $R(3,4) \leq R(2,4) + R(3,3) \leq 4 + 6 = 10$
- $R(3,5) \leq R(2,5) + R(3,4) \leq 5 + 10 = 15$
- $R(3,6) \leq R(2,6) + R(3,5) \leq 6 + 15 = 21$
- $R(3,7) \leq R(2,7) + R(3,6) \leq 7 + 21 = 28$
- $R(4,4) \leq R(3,4) + R(4,3) \leq 10 + 10 = 20$

Lets Compute Bounds on $R(a, b)$

- $R(3,3) \leq R(2,3) + R(3,2) \leq 3 + 3 = 6$
- $R(3,4) \leq R(2,4) + R(3,3) \leq 4 + 6 = 10$
- $R(3,5) \leq R(2,5) + R(3,4) \leq 5 + 10 = 15$
- $R(3,6) \leq R(2,6) + R(3,5) \leq 6 + 15 = 21$
- $R(3,7) \leq R(2,7) + R(3,6) \leq 7 + 21 = 28$
- $R(4,4) \leq R(3,4) + R(4,3) \leq 10 + 10 = 20$
- $R(4,5) \leq R(3,5) + R(4,4) \leq 15 + 20 = 35$

Lets Compute Bounds on $R(a, b)$

- $R(3,3) \leq R(2,3) + R(3,2) \leq 3 + 3 = 6$
- $R(3,4) \leq R(2,4) + R(3,3) \leq 4 + 6 = 10$
- $R(3,5) \leq R(2,5) + R(3,4) \leq 5 + 10 = 15$
- $R(3,6) \leq R(2,6) + R(3,5) \leq 6 + 15 = 21$
- $R(3,7) \leq R(2,7) + R(3,6) \leq 7 + 21 = 28$
- $R(4,4) \leq R(3,4) + R(4,3) \leq 10 + 10 = 20$
- $R(4,5) \leq R(3,5) + R(4,4) \leq 15 + 20 = 35$
- $R(5,5) \leq R(4,5) + R(5,4) \leq 35 + 35 = 70.$

Table of Bounds

$R(a,b)$	Bound on $R(a,b)$
$R(3,3)$	6
$R(3,4)$	10
$R(3,5)$	15
$R(3,6)$	21
$R(3,7)$	28
$R(4,4)$	20
$R(4,5)$	35
$R(5,5)$	70

Table of Bounds

$R(a, b)$	Bound on $R(a, b)$
$R(3, 3)$	6
$R(3, 4)$	10
$R(3, 5)$	15
$R(3, 6)$	21
$R(3, 7)$	28
$R(4, 4)$	20
$R(4, 5)$	35
$R(5, 5)$	70

Can we make some improvements to this?

Table of Bounds

$R(a,b)$	Bound on $R(a,b)$
$R(3,3)$	6
$R(3,4)$	10
$R(3,5)$	15
$R(3,6)$	21
$R(3,7)$	28
$R(4,4)$	20
$R(4,5)$	35
$R(5,5)$	70

Can we make some improvements to this? YES!

Table of Bounds

$R(a,b)$	Bound on $R(a,b)$
$R(3,3)$	6
$R(3,4)$	10
$R(3,5)$	15
$R(3,6)$	21
$R(3,7)$	28
$R(4,4)$	20
$R(4,5)$	35
$R(5,5)$	70

Can we make some improvements to this? YES!
We need a theorem.

Table of Bounds

$R(a,b)$	Bound on $R(a,b)$
$R(3,3)$	6
$R(3,4)$	10
$R(3,5)$	15
$R(3,6)$	21
$R(3,7)$	28
$R(4,4)$	20
$R(4,5)$	35
$R(5,5)$	70

Can we make some improvements to this? YES!
We need a theorem. We first do an example.

A Graph on 9 Vertices with all verts Deg 3?

Thm There is NO graph on 9 verts, with every vertex of deg 3.

A Graph on 9 Vertices with all verts Deg 3?

Thm There is NO graph on 9 verts, with every vertex of deg 3.

We count the number of edges.

A Graph on 9 Vertices with all verts Deg 3?

Thm There is NO graph on 9 verts, with every vertex of deg 3.

We count the number of edges.

Every vertex contributes 3 to the number of edges.

A Graph on 9 Vertices with all verts Deg 3?

Thm There is NO graph on 9 verts, with every vertex of deg 3.

We count the number of edges.
Every vertex contributes 3 to the number of edges.
So there are $9 \times 3 = 27$ edges.

A Graph on 9 Vertices with all verts Deg 3?

Thm There is NO graph on 9 verts, with every vertex of deg 3.

We count the number of edges.
Every vertex contributes 3 to the number of edges.
So there are $9 \times 3 = 27$ edges.
Oh. We overcounted.

A Graph on 9 Vertices with all verts Deg 3?

Thm There is NO graph on 9 verts, with every vertex of deg 3.

We count the number of edges.
Every vertex contributes 3 to the number of edges.
So there are $9 \times 3 = 27$ edges.
Oh. We overcounted. We counted every edge exactly twice.

A Graph on 9 Vertices with all verts Deg 3?

Thm There is NO graph on 9 verts, with every vertex of deg 3.

We count the number of edges.
Every vertex contributes 3 to the number of edges.
So there are $9 \times 3 = 27$ edges.
Oh. We overcounted. We counted every edge exactly twice.
Oh My! That means there are $\frac{27}{2}$ edges.

A Graph on 9 Vertices with all verts Deg 3?

Thm There is NO graph on 9 verts, with every vertex of deg 3.

We count the number of edges.
Every vertex contributes 3 to the number of edges.
So there are $9 \times 3 = 27$ edges.
Oh. We overcounted. We counted every edge exactly twice.
Oh My! That means there are $\frac{27}{2}$ edges. Contradiction.

A Graph on 9 Vertices with all verts Deg 3?

Thm There is NO graph on 9 verts, with every vertex of deg 3.

We count the number of edges.
Every vertex contributes 3 to the number of edges.
So there are $9 \times 3 = 27$ edges.
Oh. We overcounted. We counted every edge exactly twice.
Oh My! That means there are $\frac{27}{2}$ edges. Contradiction.
We generalize this on the next slide.

Handshake Lemma

Lemma Let $G = (V, E)$ be a graph.

Handshake Lemma

Lemma Let $G = (V, E)$ be a graph.

$$V_{\text{even}} = \{v : \deg(v) \equiv 0 \pmod 2\}$$
$$V_{\text{odd}} = \{v : \deg(v) \equiv 1 \pmod 2\}$$

Handshake Lemma

Lemma Let $G = (V, E)$ be a graph.

$$V_{\mathrm{even}} = \{v : \deg(v) \equiv 0 \pmod 2\}$$
$$V_{\mathrm{odd}} = \{v : \deg(v) \equiv 1 \pmod 2\}$$

Then $|V_{\mathrm{odd}}| \equiv 0 \pmod 2$.

Handshake Lemma

Lemma Let $G = (V, E)$ be a graph.

$$V_{\text{even}} = \{v : \deg(v) \equiv 0 \pmod 2\}$$
$$V_{\text{odd}} = \{v : \deg(v) \equiv 1 \pmod 2\}$$

Then $|V_{\text{odd}}| \equiv 0 \pmod 2$.

$$\sum_{v \in V_{\text{even}}} \deg(v) + \sum_{v \in V_{\text{odd}}} \deg(v) = \sum_{v \in V} \deg(v) = 2|E| \equiv 0 \pmod 2.$$

Handshake Lemma

Lemma Let $G = (V, E)$ be a graph.

$$V_{\text{even}} = \{v : \deg(v) \equiv 0 \pmod 2\}$$
$$V_{\text{odd}} = \{v : \deg(v) \equiv 1 \pmod 2\}$$

Then $|V_{\text{odd}}| \equiv 0 \pmod 2$.

$$\sum_{v \in V_{\text{even}}} \deg(v) + \sum_{v \in V_{\text{odd}}} \deg(v) = \sum_{v \in V} \deg(v) = 2|E| \equiv 0 \pmod 2.$$

$$\sum_{v \in V_{\text{odd}}} \deg(v) \equiv 0 \pmod 2.$$

Handshake Lemma

Lemma Let $G = (V, E)$ be a graph.

$$\begin{aligned} V_{\text{even}} &= \{v : \deg(v) \equiv 0 \pmod 2\} \\ V_{\text{odd}} &= \{v : \deg(v) \equiv 1 \pmod 2\} \end{aligned}$$

Then $|V_{\text{odd}}| \equiv 0 \pmod 2$.

$$\sum_{v \in V_{\text{even}}} \deg(v) + \sum_{v \in V_{\text{odd}}} \deg(v) = \sum_{v \in V} \deg(v) = 2|E| \equiv 0 \pmod 2.$$

$$\sum_{v \in V_{\text{odd}}} \deg(v) \equiv 0 \pmod 2.$$

Sum of odds $\equiv 0 \pmod 2$. Must have even numb of them. So $|V_{\text{odd}}| \equiv 0 \pmod 2$.

Handshake Lemma

Lemma Let $G = (V, E)$ be a graph.

$$V_{\mathrm{even}} = \{v : \deg(v) \equiv 0 \pmod 2\}$$
$$V_{\mathrm{odd}} = \{v : \deg(v) \equiv 1 \pmod 2\}$$

Then $|V_{\mathrm{odd}}| \equiv 0 \pmod 2$.

$$\sum_{v \in V_{\mathrm{even}}} \deg(v) + \sum_{v \in V_{\mathrm{odd}}} \deg(v) = \sum_{v \in V} \deg(v) = 2|E| \equiv 0 \pmod 2.$$

$$\sum_{v \in V_{\mathrm{odd}}} \deg(v) \equiv 0 \pmod 2.$$

Sum of odds $\equiv 0 \pmod 2$. Must have even numb of them. So $|V_{\mathrm{odd}}| \equiv 0 \pmod 2$.

Handshake Lemma If all pairs of people in a room shake hands, even number of shakes.

Handshake Lemma

Lemma Let $G = (V, E)$ be a graph.

$$V_{\rm even} = \{v : \deg(v) \equiv 0 \pmod 2\}$$
$$V_{\rm odd} = \{v : \deg(v) \equiv 1 \pmod 2\}$$

Then $|V_{\rm odd}| \equiv 0 \pmod 2$.

$$\sum_{v \in V_{\rm even}} \deg(v) + \sum_{v \in V_{\rm odd}} \deg(v) = \sum_{v \in V} \deg(v) = 2|E| \equiv 0 \pmod 2.$$

$$\sum_{v \in V_{\rm odd}} \deg(v) \equiv 0 \pmod 2.$$

Sum of odds $\equiv 0 \pmod 2$. Must have even numb of them. So $|V_{\rm odd}| \equiv 0 \pmod 2$.

Handshake Lemma If all pairs of people in a room shake hands, even number of shakes. (Pre-COVID when people shook hands.)

Corollary of Handshake Lemma

Impossible to have a graph on an odd number of verts where every vertex is of odd degree.

Corollary of Handshake Lemma

Impossible to have a graph on an odd number of verts where every vertex is of odd degree.
And NOW to our improvements on small Ramsey numbers.

$R(3,4) \leq 9$ Case 1

Assume we have a 2-coloring of the edges of K_9.

$R(3,4) \le 9$ Case 1

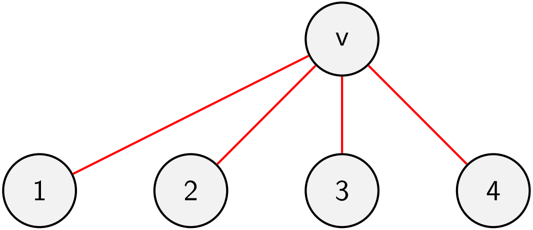
Assume we have a 2-coloring of the edges of K_9.

Case 1 $(\exists v)[\deg_R(v) \ge 4]$.

$R(3,4) \leq 9$ Case 1

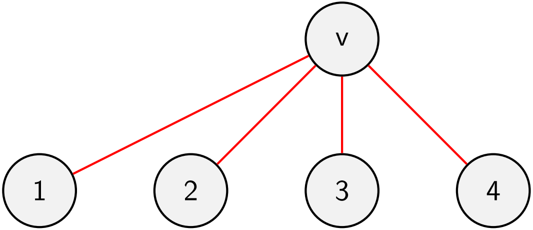
Assume we have a 2-coloring of the edges of K_9.

Case 1 $(\exists v)[\deg_R(v) \geq 4]$.



$R(3,4) \leq 9$ Case 1

Assume we have a 2-coloring of the edges of K_9.
Case 1 $(\exists v)[\deg_R(v) \geq 4]$.

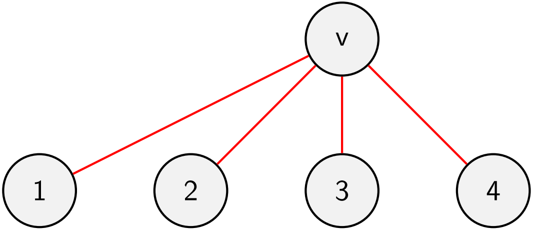


1) If **any** of $\{1,2\}$,$\{1,3\}$, $\{1,4\}$, $\{2,3\}$, $\{2,4\}$, $\{3,4\}$ are **RED**, have **RED** K_3.

$R(3,4) \leq 9$ Case 1

Assume we have a 2-coloring of the edges of K_9.

Case 1 $(\exists v)[\deg_R(v) \geq 4]$.

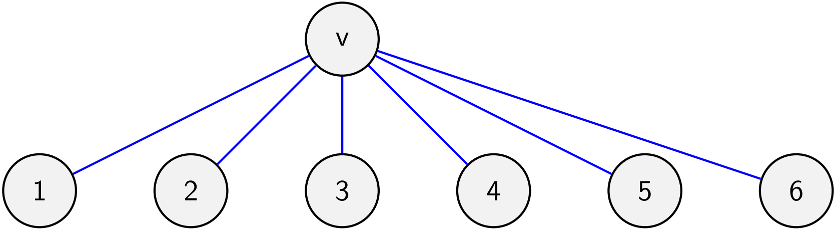


1) If **any** of $\{1,2\}$,$\{1,3\}$, $\{1,4\}$, $\{2,3\}$, $\{2,4\}$, $\{3,4\}$ are **RED**, have **RED** K_3.

2) If **all** of $\{1,2\}$,$\{1,3\}$, $\{1,4\}$, $\{2,3\}$, $\{2,4\}$, $\{3,4\}$ are **BLUE**, have **BLUE** K_4.

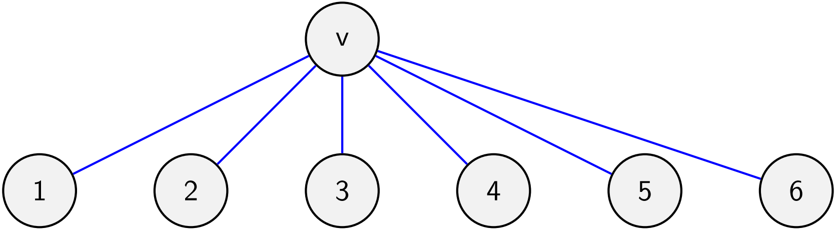
$R(3,4) \leq 9$ Case 2

Case 2 $(\exists v)[\deg_R(v) \leq 2]$, so $\deg_B(v) \geq 6$.



$R(3,4) \le 9$ Case 2

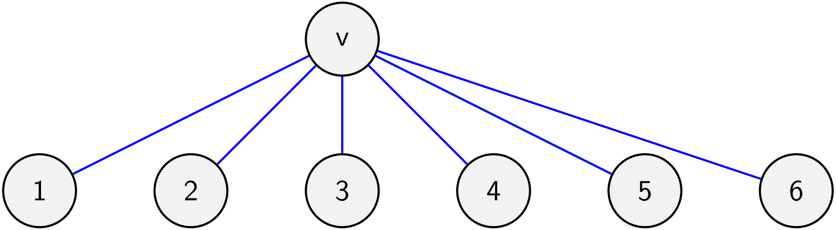
Case 2 $(\exists v)[\deg_R(v) \le 2]$, so $\deg_B(v) \ge 6$.



(1) There is a **RED** K_3 in $\{1,2,3,4,5,6\}$. Have **RED** K_3.

$R(3,4) \leq 9$ Case 2

Case 2 $(\exists v)[\deg_R(v) \leq 2]$, so $\deg_B(v) \geq 6$.



(1) There is a **RED** K_3 in $\{1,2,3,4,5,6\}$. Have **RED** K_3.

(2) There is a **BLUE** K_3. With v get a **BLUE** K_4.

$R(3,4) \leq 9$ Case 3

Recall

$R(3,4) \leq 9$ Case 3

Recall

Case 1 $(\exists v)[\deg_R(v) \geq 4]$.

$R(3,4) \leq 9$ Case 3

Recall

Case 1 $(\exists v)[\mathrm{deg}_R(v) \geq 4]$.

Case 2 $(\exists v)[\mathrm{deg}_R(v) \leq 2]$.

$R(3,4) \leq 9$ Case 3

Recall

Case 1 $(\exists v)[\deg_R(v) \geq 4]$.

Case 2 $(\exists v)[\deg_R(v) \leq 2]$.

Negation of Case 1 and Case 2 yields

$R(3,4) \le 9$ Case 3

Recall

Case 1 $(\exists v)[\deg_R(v) \ge 4]$.

Case 2 $(\exists v)[\deg_R(v) \le 2]$.

Negation of Case 1 and Case 2 yields

Case 3 $(\forall v)[\deg_R(v) = 3]$.

$R(3,4) \leq 9$ Case 3

Recall

Case 1 $(\exists v)[\deg_R(v) \geq 4]$.

Case 2 $(\exists v)[\deg_R(v) \leq 2]$.

Negation of Case 1 and Case 2 yields

Case 3 $(\forall v)[\deg_R(v) = 3]$.
SO the **RED** graph is a graph on 9 verts with all verts of degree 3.

$R(3,4) \leq 9$ Case 3

Recall

Case 1 $(\exists v)[\deg_R(v) \geq 4]$.

Case 2 $(\exists v)[\deg_R(v) \leq 2]$.

Negation of Case 1 and Case 2 yields

Case 3 $(\forall v)[\deg_R(v) = 3]$.
SO the **RED** graph is a graph on 9 verts with all verts of degree 3.
This is impossible!

A Generalization of this Trick

What was it about $R(3,4)$ that made that trick work?

A Generalization of this Trick

What was it about $R(3,4)$ that made that trick work?
We originally had

A Generalization of this Trick

What was it about $R(3,4)$ that made that trick work?
We originally had

$$R(3,4) \leq R(2,4) + R(3,3) \leq 4 + 6 \leq 10$$

A Generalization of this Trick

What was it about $R(3,4)$ that made that trick work?
We originally had

$$R(3,4) \leq R(2,4) + R(3,3) \leq 4 + 6 \leq 10$$

Key: $R(2,4)$ and $R(3,3)$ were both **even!**

A Generalization of this Trick

What was it about $R(3,4)$ that made that trick work?
We originally had

$$R(3,4) \le R(2,4) + R(3,3) \le 4 + 6 \le 10$$

Key: $R(2,4)$ and $R(3,3)$ were both **even!**

Theorem $R(a,b) \le$

1. $R(a,b-1) + R(a-1,b)$ always.
2. $R(a,b-1) + R(a-1,b) - 1$ if $R(a,b-1) \equiv R(a-1,b) \equiv 0 \pmod{2}$

A Generalization of this Trick

What was it about $R(3,4)$ that made that trick work?
We originally had

$$R(3,4) \leq R(2,4) + R(3,3) \leq 4 + 6 \leq 10$$

Key: $R(2,4)$ and $R(3,3)$ were both **even!**

Theorem $R(a,b) \leq$

1. $R(a,b-1) + R(a-1,b)$ always.
2. $R(a,b-1) + R(a-1,b) - 1$ if $R(a,b-1) \equiv R(a-1,b) \equiv 0 \pmod 2$

Proof left to the Reader.

Some Better Upper Bounds

- $R(3,3) \leq R(2,3) + R(3,2) \leq 3 + 3 = 6.$
- $R(3,4) \leq R(2,4) + R(3,3) \leq 4 + 6 - 1 = 9.$
- $R(3,5) \leq R(2,5) + R(3,4) \leq 5 + 9 = 14.$
- $R(3,6) \leq R(2,6) + R(3,5) \leq 6 + 14 - 1 = 19.$
- $R(3,7) \leq R(2,7) + R(3,6) \leq 7 + 19 = 26$
- $R(4,4) \leq R(3,4) + R(4,3) \leq 9 + 9 = 18.$
- $R(4,5) \leq R(3,5) + R(4,4) \leq 14 + 18 - 1 = 31.$
- $R(5,5) \leq R(4,5) + R(5,4) = 62.$

Some Better Upper Bounds

- $R(3,3) \leq R(2,3) + R(3,2) \leq 3 + 3 = 6.$
- $R(3,4) \leq R(2,4) + R(3,3) \leq 4 + 6 - 1 = 9.$
- $R(3,5) \leq R(2,5) + R(3,4) \leq 5 + 9 = 14.$
- $R(3,6) \leq R(2,6) + R(3,5) \leq 6 + 14 - 1 = 19.$
- $R(3,7) \leq R(2,7) + R(3,6) \leq 7 + 19 = 26$
- $R(4,4) \leq R(3,4) + R(4,3) \leq 9 + 9 = 18.$
- $R(4,5) \leq R(3,5) + R(4,4) \leq 14 + 18 - 1 = 31.$
- $R(5,5) \leq R(4,5) + R(5,4) = 62.$

There have been some improvements, including one I will have you read about.