CMSC 250H - Midterm Exam Slutions

April 1, 2026

Question 2

Question

Find $X \in \mathbb{N}$ such that the following is true:

For all $n \geq X$, there is a boolean formula with n variables that is satisfied by exactly $n^2 + n$ assignments.

2 – Solution

- We know that there are 2^n possible assignments for any boolean formula on n variables.
- We can construct a formula with a certain number of satisfying assignments
- So all we need is $2^n \geq n^2 + n$

Solution

$$n = 2 \longrightarrow 2^2 \ngeq 2^2 + 2$$
$$n = 3 \longrightarrow 2^3 \ngeq 3^2 + 3$$
$$n = 4 \longrightarrow 2^4 \ngeq 4^2 + 4$$
$$n = 5 \longrightarrow 2^5 \geq 5^2 + 5$$

So $X = 5$

Question 3

Question

Use Unique Factorization to prove the following:

$$\text{Let } n = p_1^{n_1} p_2^{n_2} \ldots p_k^{n_k}, \text{ where } p_1, p_2, \ldots, p_k \text{ are primes.}$$
$$\text{If } n^{1/3} \text{ is rational, then } (\forall 1 \leq i \leq k)[n_i \equiv 0 \mod 3]$$

3 – Solution

- Our UFP proofs so far have been formulaic.
- If we have a rational number $x = \frac{a}{b}$, We always consider the prime factorizations of a and b
- Only difference with this proof is we do it directly instead of using contradiction

3 – Solution

Proof.

Suppose $n^{1/3}$ is rational. Then we can write

$$n^{1/3} = \frac{a}{b}$$

with $a \in \mathbb{Z}$ and $b \in \mathbb{Z} \setminus \{0\}$. So, we have that

$$n = \frac{a^3}{b^3} \implies n \cdot b^3 = a^3$$

Now, let $q_1 \ldots q_m$ denote the prime factors of either a or b, and let the prime factorizations of a and b be

$$a = q_1^{a_1} q_2^{a_2} \ldots q_m^{a_m}$$
$$b = q_1^{b_1} q_2^{b_2} \ldots q_m^{b_m}$$

Proof.

Plugging in our UFP's, we obtain

$$(p_1^{n_1} p_2^{n_2} \dots p_k^{n_k})(q_1^{3b_1} q_2^{3b_2} \dots q_m^{3b_m}) = q_1^{3a_1} q_2^{3a_2} \dots q_m^{3a_m}$$

Clearly all q_i have exponents $0 \mod 3$. Consider the case where $p_i = q_j$ for some $1 \leq i \leq k$ and $1 \leq j \leq m$. The exponent for this term on the LHS now becomes

$$3b_j + n_i$$

Since the exponent of q_j on the RHS is $3a_j$, it must be that

$$3b_j + n_i = 3a_j \implies n_i = 3(a_j - b_j)$$

Hence $n_i \equiv 0 \mod 3$ for all i. □

Problem 3 – Grading

- If you got up to plugging in the UFP's for $n \longrightarrow$ +5 points
- If you got up to plugging in and considering UFP's for all a, b, and $n \longrightarrow$ +10 points
- If you got all of that right, and then explained the reasoning at the end with english (but it made sense) $\longrightarrow$ +15 points

Question 4

Question 4– a)

Compute the following $\mod 9$:

$$0^6, 1^6, 2^6, \ldots, 8^6$$

You may use the fact $(9 - a)^6 \equiv a^6 \mod 9$

4a) – Solution

Solution

$$0^6 \equiv 0 \mod 9$$

$$1^6 \equiv 1 \mod 9$$

$$2^6 \equiv 64 \equiv 1 \mod 9$$

$$3^6 \equiv 0 \mod 9$$

$$4^6 \equiv (4^3)^2 \equiv 1 \mod 9$$

$$5^6 \equiv 1 \mod 9$$

$$6^6 \equiv 0 \mod 9$$

$$7^6 \equiv 1 \mod 9$$

$$8^6 \equiv 1 \mod 9$$

Question 4 – b)

Use the results of part a to find a number N and an infinite set X such that the following is true:

If $x \in X$, then x cannot be written as the sum of N sixth powers. Make N as large as possible.

4b) – Solution

- We had one just like this on the Practice Exam
- Since the only 6^{th} powers $\mod 9$ are 0 and 1, we can only add up 1's a certain number of times. So, if I have 8 ones, then I can write any number mod 9 using 8 ones.
- If I only have 7 ones, then I can't write a number $\equiv 8 \mod 9$ as a sum of 7 6^{th} powers. So we have:

Solution

$$N = 7$$

and

$$X = \{x : x \equiv 8 \mod 9\}$$

Question 5

Question

Let a_n be defined as follows:

$$\begin{aligned} a_0 &= 11 \\ a_1 &= 21 \\ (\forall n \geq 2)[a_n &= 7a_{n-1} + 4a_{n-2}] \end{aligned}$$

Prove that

$$(\forall n \geq 0)[a_n \equiv 1 \mod 10]$$

5 – Solution

Proof.

Base Case: We have for $n = 0$ and $n = 1$

$$a_0 = 11 = 10 + 1 \equiv 1 \mod 10$$
$$a_1 = 21 = 20 + 1 \equiv 1 \mod 10$$

So the statement holds for $n = 0, 1$.

Induction Hypothesis: Suppose that $a_k \equiv 1 \mod 10$ for k with $0 \leq k < n$. □

Proof.

We need to show $a_{k+1} \equiv 1 \mod 10$ using our hypothesis.

Inductive Step: We have that

$$a_{k+1} = 7a_k + 4a_{k-1}$$

By the IH, we can know $a_k \equiv a_{k-1} \equiv 1 \mod 10$. So we can write

$$a_k = 10a + 1 \qquad a_{k-1} = 10b + 1$$

for $a, b \in \mathbb{Z}$ □

5 – Solution

Proof.

Plugging in

$$\begin{aligned} a_{k+1} &= 7(10a+1) + 4(10b+1) \\ &= 70a + 7 + 40b + 4 \\ &= 10(7a) + 10(4b) + 11 \\ &= 11 \mod 10 \\ &= 1 \mod 10 \end{aligned}$$

as desired. □

Question 5 Grading

- Base case + hypothesis $\longrightarrow$ +5 points
- Induction step correct, then didn't care about missing the other stuff $\longrightarrow$ Full Credit
- For a lot of you, we gave full credit, but doing similar mistakes on the final won't get you points.

Question 6

Here, the domains are $\subseteq \mathbb{R}$. Consider the following sentence:

$$(\forall x)(\exists y)[x + y = \sqrt{2}]$$

a) Do one of the following:
   - Give an infinite domain where A is true.
   - State that there is no such domain.

b) Do one of the following:
   - Give a finite domain with at least 2 elements where A is true.
   - State that there is no such domain.

6 – solution

- For the statement to hold, we need $x + y = \sqrt{2} \implies y = \sqrt{2} - x$
- So if $x \in D$, then $\sqrt{2} - x$ should also be in D.

a) Solution

The real numbers

$$D = \mathbb{R}$$

satisfies this.

b) Solution

The set

$$D = \{0, \sqrt{2}, \sqrt{2} - 1, 1\}$$

is an example of a finite set that works