The $x^n + y^n$ Theorem

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Example that uses Notation!

$(\forall n \in \mathbb{N})(\exists x, y \in \mathbb{R})$
$[(x + y \in \mathbb{Z}) \wedge (x^n + y^n \in \mathbb{Z}) \wedge (x^{n+1} + y^{n+1} \notin \mathbb{Z})]$.

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For more terminology of this type see my blog post:
`https://blog.computationalcomplexity.org/2023/08/theorems-and-lemmas-and-proofs-oh-my.html`

The $(x + y, x^2 + y^2)$ Problem

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Discuss. Answer on next slide.

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$\left(\frac{3+\sqrt{23}}{2}\right)^3 + \left(\frac{3-\sqrt{23}}{2}\right)^3 \notin \mathbb{Z}.$
(We leave that one to you)

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Yes. $k = 4$ works.

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Yes. $k = 4$ works. We prove this which is our main theorem.

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$2a^2 = 2(\frac{b}{2})^2 = 2(\frac{b^2}{4}) = \frac{b^2}{2} = \frac{4c^2+4c+1}{4} \notin \mathbb{Z}$.

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Since $2(xy) \in \mathbb{Z}$ and $2(x^2y^2) \in \mathbb{Z}$,

by FIRST LEMMA, $xy \in \mathbb{Z}$.

A Small Aside For A Review of Induction

Review of Induction Part I

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We have $P(2)$ and $P(2) \rightarrow P(3)$. So we have $P(3)$.
... We can keep doing this until we get up to $P(n)$.

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We will prove
$P(1)$, $P(2)$, $P(3)$, $P(4)$.
$(\forall n \geq 4)[(P(n-1) \wedge P(n)) \rightarrow P(n+1)]$.
I claim that we have proven $(\forall n)[P(n)]$.

We have $P(3), P(4)$ and $P(3) \wedge P(4) \rightarrow P(5)$. So we have $P(5)$.
We have $P(4), P(5)$ and $P(4) \wedge P(5) \rightarrow P(6)$. So we have $P(6)$.
... We can keep doing this until we get up to $P(n)$.

Back to Our Theorem

$x+y, \ldots, x^4+y^4$ Theorem

$x + y, \ldots, x^4 + y^4$ Theorem

Thm Let $x, y \in \mathbb{R}$.

$x + y, \ldots, x^4 + y^4$ Theorem

Thm Let $x, y \in \mathbb{R}$. Assume the following are in $\mathbb{Z}$:
$x + y$, $x^2 + y^2$, $x^3 + y^3$, $x^4 + y^4$.

$x + y, \ldots, x^4 + y^4$ Theorem

Thm Let $x, y \in \mathbb{R}$. Assume the following are in $\mathbb{Z}$:
$x + y$, $x^2 + y^2$, $x^3 + y^3$, $x^4 + y^4$.
Then $(\forall n \in \mathbb{N})[x^n + y^n \in \mathbb{Z}]$.

$x + y, \ldots, x^4 + y^4$ Theorem

Thm Let $x, y \in \mathbb{R}$. Assume the following are in $\mathbb{Z}$:
$x + y$, $x^2 + y^2$, $x^3 + y^3$, $x^4 + y^4$.
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Base $n = 1, 2, 3, 4$ are given.

$x + y, \ldots, x^4 + y^4$ Theorem

Thm Let $x, y \in \mathbb{R}$. Assume the following are in $\mathbb{Z}$:
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Base $n = 1, 2, 3, 4$ are given.

IH Thm true for n and $n + 1$. (We prove for $n + 2$.)

$x + y, \ldots, x^4 + y^4$ Theorem

Thm Let $x, y \in \mathbb{R}$. Assume the following are in $\mathbb{Z}$:
$x + y$, $x^2 + y^2$, $x^3 + y^3$, $x^4 + y^4$.
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IH Thm true for n and $n + 1$. (We prove for $n + 2$.)

IS $x^{n+2} + y^{n+2} = (x + y)(x^{n+1} + y^{n+1}) - xy(x^n + y^n)$.

$x+y,\ldots,x^4+y^4$ Theorem

Thm Let $x, y \in \mathbb{R}$. Assume the following are in $\mathbb{Z}$:
$x+y$, x^2+y^2, x^3+y^3, x^4+y^4.
Then $(\forall n \in \mathbb{N})[x^n + y^n \in \mathbb{Z}]$.

Base $n = 1, 2, 3, 4$ are given.

IH Thm true for n and $n+1$. (We prove for $n+2$.)

IS $x^{n+2} + y^{n+2} = (x+y)(x^{n+1} + y^{n+1}) - xy(x^n + y^n)$.

By the premise $x + y \in \mathbb{Z}$.

$x + y, \ldots, x^4 + y^4$ Theorem

Thm Let $x, y \in \mathbb{R}$. Assume the following are in $\mathbb{Z}$:
$x + y$, $x^2 + y^2$, $x^3 + y^3$, $x^4 + y^4$.
Then $(\forall n \in \mathbb{N})[x^n + y^n \in \mathbb{Z}]$.

Base $n = 1, 2, 3, 4$ are given.

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IS $x^{n+2} + y^{n+2} = (x + y)(x^{n+1} + y^{n+1}) - xy(x^n + y^n)$.

By the premise $x + y \in \mathbb{Z}$.

By the IH $x^{n+1} + y^{n+1} \in \mathbb{Z}$ and $x^n + y^n \in \mathbb{Z}$.

$x + y, \ldots, x^4 + y^4$ Theorem

Thm Let $x, y \in \mathbb{R}$. Assume the following are in $\mathbb{Z}$:
$x + y$, $x^2 + y^2$, $x^3 + y^3$, $x^4 + y^4$.
Then $(\forall n \in \mathbb{N})[x^n + y^n \in \mathbb{Z}]$.

Base $n = 1, 2, 3, 4$ are given.

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By the premise $x + y \in \mathbb{Z}$.

By the IH $x^{n+1} + y^{n+1} \in \mathbb{Z}$ and $x^n + y^n \in \mathbb{Z}$.

By SECOND LEMMA $xy \in \mathbb{Z}$.

$x+y,\ldots,x^4+y^4$ Theorem

Thm Let $x, y \in \mathbb{R}$. Assume the following are in $\mathbb{Z}$:
$x+y$, x^2+y^2, x^3+y^3, x^4+y^4.
Then $(\forall n \in \mathbb{N})[x^n + y^n \in \mathbb{Z}]$.

Base $n = 1, 2, 3, 4$ are given.

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IS $x^{n+2} + y^{n+2} = (x+y)(x^{n+1} + y^{n+1}) - xy(x^n + y^n)$.

By the premise $x + y \in \mathbb{Z}$.

By the IH $x^{n+1} + y^{n+1} \in \mathbb{Z}$ and $x^n + y^n \in \mathbb{Z}$.

By SECOND LEMMA $xy \in \mathbb{Z}$.

Hence
$x^{n+2} + y^{n+2} = (x+y)(x^{n+1} + y^{n+1}) - xy(x^n + y^n) \in \mathbb{Z}$.