Loaded Dice

William Gasarch - University of MD

If You Roll Two Standard 6-Sided Dice Then

1. 2: (1,1). ONE way. Prob $\frac{1}{36}$.
2. 3: (1,2), (2,1). TWO ways. Prob $\frac{1}{18}$.
3. 4: (1,3), (2,2), (3,1). THREE ways. Prob $\frac{1}{12}$.
4. 5: (1,4), (2,3), (3,2), (4,1). FOUR ways. Prob $\frac{1}{9}$.
5. 6: (1,5), (2,4), (3,3), (4,2), (5,1) FIVE ways. Prob $\frac{5}{36}$.

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6. 7: (1,6), (2,5), (3,4), (4,3), (5,2), (6,1) SIX ways. Prob $\frac{1}{6}$.

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8. 9: (3,6), (4,5), (5,4), (6,3) FOUR ways. Prob $\frac{1}{9}$.
9. 10: (4,6), (5,5), (6,4) THREE ways. Prob $\frac{1}{12}$.
10. 11: (5,6), (6,5) TWO ways. Prob $\frac{1}{18}$.
11. 12: (6,6) ONE way. Prob $\frac{1}{36}$.

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Question 1 Can we load two 6-sided dice so that every number from 2 to 12 has the **same** probability. Called **fair sums**.

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How Unfair?: 1/6 - 1/36 $\sim$ 0.139 unfair.

What Are Loaded Dice?

Def: A **Die** is a 6-tuple $(p_1, p_2, p_3, p_4, p_5, p_6)$ such that $0 \leq p_i \leq 1$ and $\sum_{i=1}^{6} p_i = 1$.

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3. Answer on next slide.

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The coefficient of x^i is $\mathrm{Prob}(\mathrm{sum} = \mathrm{i})$

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Continued on Next Slide.

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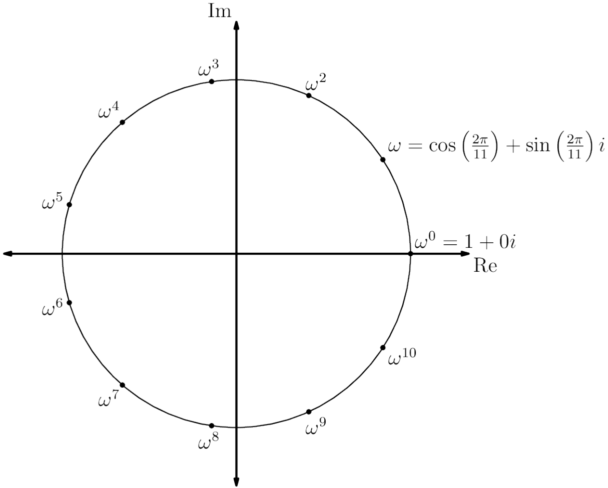
The roots of $x^{11} - 1$ are on the complex unit circle. See Next Slide.

The 11th Roots of Unity: Only Real one is 1



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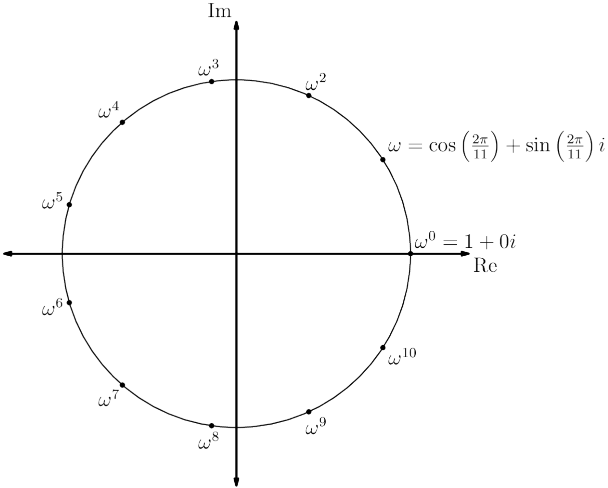

1 is only real 11th root of unity. $x^{10} + \cdots + 1 = 0$: **no** real roots.

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Recap

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Contradiction

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Can You Ever Load Dice to Get Fair Sums?

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Some people say that using 0 as a probability should not be allowed.

We turn that objection into a math question:

Is there a $d_1, d_2 \geq 2$ such that d_1-sided and d_2-sided dice that give fair sums, with all the probs on the dice > 0?
VOTE: YES or NO or UNKNOWN TO SCIENCE!
Answer on Next Slide.

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Fame! One paper refers to **The Gasarch-Kruskal Thm**.

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How far are normal dice from uniform?

$$2(1/11-1/36)^2+2(1/11-1/18)^2+2(1/11-1/12)^2+2(1/9-1/11)^2+$$

$$2(5/36-1/11)^2)+(1/6-1/11)^2 \sim 0.0217$$

How Close To Uniform Can You Get? (cont)

Thm The optimal pair of 6-sided dice is $(\frac{1}{2}, 0, 0, 0, 0, \frac{1}{2})$ and $(\frac{1}{8}, \frac{3}{16}, \frac{3}{16}, \frac{3}{16}, \frac{3}{16}, \frac{1}{8})$.

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The distance from uniform is $\frac{1}{2(2n-1)(3n-2)} \sim \frac{1}{12n^2}$.

If you use two n-sided fair dice then the the distance from uniform sums is $\frac{2n^2+1}{3n^3} - \frac{1}{2n-1} \sim \frac{1}{6n}$.