Sicherman Dice

William Gasarch - University of MD

Different Labels on Dice

William Gasarch - University of MD

Can You Label Dice To Get Same Probs?

A **labeling** of a 6-sided die has any positive natural numbers as labels. We allow using a number twice. We allow using numbers higher than 6. So $(1, 2, 2, 3, 5, 8)$ would be allowed.

Can You Label Dice To Get Same Probs?

A **labeling** of a 6-sided die has any positive natural numbers as labels. We allow using a number twice. We allow using numbers higher than 6. So $(1, 2, 2, 3, 5, 8)$ would be allowed.

A **non-standard labeling** is a labeling that is **not** $(1, 2, 3, 4, 5, 6)$.

Can You Label Dice To Get Same Probs?

A **labeling** of a 6-sided die has any positive natural numbers as labels. We allow using a number twice. We allow using numbers higher than 6. So $(1, 2, 2, 3, 5, 8)$ would be allowed.

A **non-standard labeling** is a labeling that is **not** $(1, 2, 3, 4, 5, 6)$.

Is there a non-standard labeling of a pair of 6-sided dice so that the dice yield the **same** probabilities as the standard dice?

Can You Label Dice To Get Same Probs?

A **labeling** of a 6-sided die has any positive natural numbers as labels. We allow using a number twice. We allow using numbers higher than 6. So $(1, 2, 2, 3, 5, 8)$ would be allowed.

A **non-standard labeling** is a labeling that is **not** $(1, 2, 3, 4, 5, 6)$.

Is there a non-standard labeling of a pair of 6-sided dice so that the dice yield the **same** probabilities as the standard dice?
VOTE: YES or NO or UNKNOWN TO SCIENCE!

Can You Label Dice To Get Same Probs?

A **labeling** of a 6-sided die has any positive natural numbers as labels. We allow using a number twice. We allow using numbers higher than 6. So $(1, 2, 2, 3, 5, 8)$ would be allowed.

A **non-standard labeling** is a labeling that is **not** $(1, 2, 3, 4, 5, 6)$.

Is there a non-standard labeling of a pair of 6-sided dice so that the dice yield the **same** probabilities as the standard dice?
VOTE: YES or NO or UNKNOWN TO SCIENCE!
Answer on next slide.

Yes We Can!

YES. There a non-standard labeling of a pair of 6-sided dice so that the dice yield the SAME probabilities as the standard dice.

Yes We Can!

YES. There a non-standard labeling of a pair of 6-sided dice so that the dice yield the SAME probabilities as the standard dice.

We prove this on the next slide.

Let Polynomials Do The Work For You!

$$(x^6 + x^5 + x^4 + x^3 + x^2 + x)(x^6 + x^5 + x^4 + x^3 + x^2 + x)$$

Let Polynomials Do The Work For You!

$$(x^6 + x^5 + x^4 + x^3 + x^2 + x)(x^6 + x^5 + x^4 + x^3 + x^2 + x)$$

Look at coefficient of x^6

Let Polynomials Do The Work For You!

$$(x^6 + x^5 + x^4 + x^3 + x^2 + x)(x^6 + x^5 + x^4 + x^3 + x^2 + x)$$

Look at coefficient of x^6

$$x^1x^5 + x^2x^4 + x^3x^3 + x^4x^2 + x^5x^1 = 5x^6$$

Let Polynomials Do The Work For You!

$$(x^6 + x^5 + x^4 + x^3 + x^2 + x)(x^6 + x^5 + x^4 + x^3 + x^2 + x)$$

Look at coefficient of x^6

$$x^1x^5 + x^2x^4 + x^3x^3 + x^4x^2 + x^5x^1 = 5x^6$$

$$= (\text{Number of ways to get 6})x^6$$

Let Polynomials Do The Work For You!

$$(x^6 + x^5 + x^4 + x^3 + x^2 + x)(x^6 + x^5 + x^4 + x^3 + x^2 + x)$$

Look at coefficient of x^6

$$x^1x^5 + x^2x^4 + x^3x^3 + x^4x^2 + x^5x^1 = 5x^6$$

$$= (\text{Number of ways to get } 6)x^6$$

Coefficient of x^n is number of ways to get n.

Example of Non-Standard Labelings

Label dice $(1, 2_1, 2_2, 2_3, 5_1, 5_2)$ and $(1, 3_1, 3_2, 3_3, 3_4, 7)$?

Example of Non-Standard Labelings

Label dice $(1, 2_1, 2_2, 2_3, 5_1, 5_2)$ and $(1, 3_1, 3_2, 3_3, 3_4, 7)$?

$$(2x^5+3x^2+x)(x^7+4x^3+x) = 2x^{12}+3x^9+9x^8+2x^6+12x^5+4x^4+3x^3+x^2$$

Example of Non-Standard Labelings

Label dice $(1, 2_1, 2_2, 2_3, 5_1, 5_2)$ and $(1, 3_1, 3_2, 3_3, 3_4, 7)$?

$$(2x^5+3x^2+x)(x^7+4x^3+x) = 2x^{12}+3x^9+9x^8+2x^6+12x^5+4x^4+3x^3+x^2$$

Coefficient of x^8: $2x^5 \times 4x^3 + x \times x^7 = 9x^8$.

Example of Non-Standard Labelings

Label dice $(1, 2_1, 2_2, 2_3, 5_1, 5_2)$ and $(1, 3_1, 3_2, 3_3, 3_4, 7)$?

$$(2x^5+3x^2+x)(x^7+4x^3+x) = 2x^{12}+3x^9+9x^8+2x^6+12x^5+4x^4+3x^3+x^2$$

Coefficient of x^8: $2x^5 \times 4x^3 + x \times x^7 = 9x^8$.

How many ways can I roll an 8?

Example of Non-Standard Labelings

Label dice $(1, 2_1, 2_2, 2_3, 5_1, 5_2)$ and $(1, 3_1, 3_2, 3_3, 3_4, 7)$?

$$(2x^5+3x^2+x)(x^7+4x^3+x) = 2x^{12}+3x^9+9x^8+2x^6+12x^5+4x^4+3x^3+x^2$$

Coefficient of x^8: $2x^5 \times 4x^3 + x \times x^7 = 9x^8$.

How many ways can I roll an 8?

$(5_1, 3_1)$, $(5_1, 3_2)$, $(5_1, 3_3)$, $(5_1, 3_4)$, $(5_2, 3_1)$, $(5_2, 3_2)$, $(5_2, 3_3)$, $(5_2, 3_4)$, $(1, 7)$.

Example of Non-Standard Labelings

Label dice $(1, 2_1, 2_2, 2_3, 5_1, 5_2)$ and $(1, 3_1, 3_2, 3_3, 3_4, 7)$?

$(2x^5+3x^2+x)(x^7+4x^3+x) = 2x^{12}+3x^9+9x^8+2x^6+12x^5+4x^4+3x^3+x^2$

Coefficient of x^8: $2x^5 \times 4x^3 + x \times x^7 = 9x^8$.
How many ways can I roll an 8?
$(5_1, 3_1)$, $(5_1, 3_2)$, $(5_1, 3_3)$, $(5_1, 3_4)$, $(5_2, 3_1)$, $(5_2, 3_2)$, $(5_2, 3_3)$, $(5_2, 3_4)$, $(1, 7)$. 9 ways.

Example of Non-Standard Labelings

Label dice $(1, 2_1, 2_2, 2_3, 5_1, 5_2)$ and $(1, 3_1, 3_2, 3_3, 3_4, 7)$?

$$(2x^5+3x^2+x)(x^7+4x^3+x) = 2x^{12}+3x^9+9x^8+2x^6+12x^5+4x^4+3x^3+x^2$$

Coefficient of x^8: $2x^5 \times 4x^3 + x \times x^7 = 9x^8$.
How many ways can I roll an 8?
$(5_1, 3_1)$, $(5_1, 3_2)$, $(5_1, 3_3)$, $(5_1, 3_4)$, $(5_2, 3_1)$, $(5_2, 3_2)$, $(5_2, 3_3)$, $(5_2, 3_4)$, $(1, 7)$. 9 ways.
The coeff of x^8.

Example of Non-Standard Labelings

Label dice $(1, 2_1, 2_2, 2_3, 5_1, 5_2)$ and $(1, 3_1, 3_2, 3_3, 3_4, 7)$?

$$(2x^5+3x^2+x)(x^7+4x^3+x) = 2x^{12}+3x^9+9x^8+2x^6+12x^5+4x^4+3x^3+x^2$$

Coefficient of x^8: $2x^5 \times 4x^3 + x \times x^7 = 9x^8$.
How many ways can I roll an 8?
$(5_1, 3_1)$, $(5_1, 3_2)$, $(5_1, 3_3)$, $(5_1, 3_4)$, $(5_2, 3_1)$, $(5_2, 3_2)$, $(5_2, 3_3)$, $(5_2, 3_4)$, $(1, 7)$. 9 ways.
The coeff of x^8. coeff of x^n is numb ways to get n.

Example of Non-Standard Labelings

Label dice $(1, 2_1, 2_2, 2_3, 5_1, 5_2)$ and $(1, 3_1, 3_2, 3_3, 3_4, 7)$?

$$(2x^5+3x^2+x)(x^7+4x^3+x) = 2x^{12}+3x^9+9x^8+2x^6+12x^5+4x^4+3x^3+x^2$$

Coefficient of x^8: $2x^5 \times 4x^3 + x \times x^7 = 9x^8$.
How many ways can I roll an 8?
$(5_1, 3_1)$, $(5_1, 3_2)$, $(5_1, 3_3)$, $(5_1, 3_4)$, $(5_2, 3_1)$, $(5_2, 3_2)$, $(5_2, 3_3)$, $(5_2, 3_4)$, $(1, 7)$. 9 ways.
The coeff of x^8. coeff of x^n is numb ways to get n.

Example of Non-Standard Labelings

Label dice $(1, 2_1, 2_2, 2_3, 5_1, 5_2)$ and $(1, 3_1, 3_2, 3_3, 3_4, 7)$?

$$(2x^5+3x^2+x)(x^7+4x^3+x) = 2x^{12}+3x^9+9x^8+2x^6+12x^5+4x^4+3x^3+x^2$$

Coefficient of x^8: $2x^5 \times 4x^3 + x \times x^7 = 9x^8$.
How many ways can I roll an 8?
$(5_1, 3_1)$, $(5_1, 3_2)$, $(5_1, 3_3)$, $(5_1, 3_4)$, $(5_2, 3_1)$, $(5_2, 3_2)$, $(5_2, 3_3)$, $(5_2, 3_4)$, $(1, 7)$. 9 ways.
The coeff of x^8. coeff of x^n is numb ways to get n.

1. 12: TWO ways. Prob $\frac{1}{18}$.
2. 9: THREE ways. Prob $\frac{1}{12}$.
3. 8: NINE ways. Prob $\frac{1}{4}$.
4. 6: TWO ways. Prob $\frac{1}{18}$.

Example of Non-Standard Labelings

Label dice $(1, 2_1, 2_2, 2_3, 5_1, 5_2)$ and $(1, 3_1, 3_2, 3_3, 3_4, 7)$?

$(2x^5+3x^2+x)(x^7+4x^3+x) = 2x^{12}+3x^9+9x^8+2x^6+12x^5+4x^4+3x^3+x^2$

Coefficient of x^8: $2x^5 \times 4x^3 + x \times x^7 = 9x^8$.
How many ways can I roll an 8?
$(5_1, 3_1)$, $(5_1, 3_2)$, $(5_1, 3_3)$, $(5_1, 3_4)$, $(5_2, 3_1)$, $(5_2, 3_2)$, $(5_2, 3_3)$, $(5_2, 3_4)$, $(1, 7)$. 9 ways.
The coeff of x^8. coeff of x^n is numb ways to get n.

1. 12: TWO ways. Prob $\frac{1}{18}$.
2. 9: THREE ways. Prob $\frac{1}{12}$.
3. 8: NINE ways. Prob $\frac{1}{4}$.
4. 6: TWO ways. Prob $\frac{1}{18}$.
5. 5: TWELVE ways. Prob $\frac{1}{3}$.
6. 4: FOUR ways. Prob $\frac{1}{9}$.
7. 3: THREE ways. Prob $\frac{1}{12}$.
8. 2: ONE ways. Prob $\frac{1}{36}$.

Is there a Non-Standard Labeling That. . .

Question Is there a **nonstandard labeling** of two 6-sided dice that gives the same probabilities as the standard dice?

Is there a Non-Standard Labeling That...

Question Is there a **nonstandard labeling** of two 6-sided dice that gives the same probabilities as the standard dice?

Question Phrased In Terms of Polynomials Does there exist $a_1 \geq \cdots \geq a_6$ and $b_1 \geq \cdots \geq b_6$ such that

Is there a Non-Standard Labeling That...

Question Is there a **nonstandard labeling** of two 6-sided dice that gives the same probabilities as the standard dice?
Question Phrased In Terms of Polynomials Does there exist $a_1 \geq \cdots \geq a_6$ and $b_1 \geq \cdots \geq b_6$ such that

$$(x^{a_1}+x^{a_2}+x^{a_3}+x^{a_4}+x^{a_5}+x^{a_6})(x^{b_1}+x^{b_2}+x^{b_3}+x^{b_4}+x^{b_5}+x^{b_6}) =$$

$$(x^6+x^5+x^4+x^3+x^2+x)(x^6+x^5+x^4+x^3+x^2+x).$$

Is there a Non-Standard Labeling That... Cont.

$$(x^{a_1}+x^{a_2}+x^{a_3}+x^{a_4}+x^{a_5}+x^{a_6})(x^{b_1}+x^{b_2}+x^{b_3}+x^{b_4}+x^{b_5}+x^{b_6}) =$$

$$(x^6+x^5+x^4+x^3+x^2+x)^2 = x^2(x^5+x^4+x^3+x^2+x+1) =$$

$$x^2(x+1)^2(x^2-x+1)^2(x^2+x+1)^2.$$

Need to Factor...

Need to factor

$$x^2(x+1)^2(x^2-x+1)^2(x^2+x+1)^2.$$

into two polynomials, each of which represents a 6-sided die.

Need to Factor. . .

Need to factor

$$x^2(x+1)^2(x^2-x+1)^2(x^2+x+1)^2.$$

into two polynomials, each of which represents a 6-sided die.

Finite Number of cases.

Need to Factor...

Need to factor

$$x^2(x+1)^2(x^2-x+1)^2(x^2+x+1)^2.$$

into two polynomials, each of which represents a 6-sided die.

Finite Number of cases.

$x(x+1)(x^2+x+1)$ * $x(x+1)(x^2-x+1)^2(x^2+x+1)$

Need to Factor...

Need to factor

$$x^2(x+1)^2(x^2-x+1)^2(x^2+x+1)^2.$$

into two polynomials, each of which represents a 6-sided die.

Finite Number of cases.

$x(x+1)(x^2+x+1)$ * $x(x+1)(x^2-x+1)^2(x^2+x+1)$

$x(x+1)(x^2+x+1) = x^4+2x^3+2x^2+x.$
DIE: $(1,2,2,3,3,4)$

Need to Factor...

Need to factor

$$x^2(x+1)^2(x^2-x+1)^2(x^2+x+1)^2.$$

into two polynomials, each of which represents a 6-sided die.

Finite Number of cases.

$x(x+1)(x^2+x+1) * x(x+1)(x^2-x+1)^2(x^2+x+1)$

$x(x+1)(x^2+x+1) = x^4+2x^3+2x^2+x$.
DIE: $(1,2,2,3,3,4)$

$x(x+1)(x^2-x+1)^2(x^2+x+1) = x^8+x^6+x^5+x^4+x^3+x$.
DIE: $(1,3,4,5,6,8)$.

Need to Factor...

Need to factor

$$x^2(x+1)^2(x^2-x+1)^2(x^2+x+1)^2.$$

into two polynomials, each of which represents a 6-sided die.

Finite Number of cases.

$x(x+1)(x^2+x+1)$ * $x(x+1)(x^2-x+1)^2(x^2+x+1)$

$x(x+1)(x^2+x+1) = x^4+2x^3+2x^2+x.$
DIE: $(1,2,2,3,3,4)$

$x(x+1)(x^2-x+1)^2(x^2+x+1) = x^8+x^6+x^5+x^4+x^3+x.$
DIE: $(1,3,4,5,6,8)$.

So desired dice are $(1,2,2,3,3,4)$ and $(1,3,4,5,6,8)$.

Any Other Non-Standard Labelings?

No.

Any Other Non-Standard Labelings?

No.
Every way to factor

$$x(x+1)(x^2-x+1)^2(x^2+x+1) = x^8+x^6+x^5+x^4+x^3+x.$$

into two polys with coeff-sum 6 is either

Any Other Non-Standard Labelings?

No.
Every way to factor

$$x(x+1)(x^2-x+1)^2(x^2+x+1) = x^8+x^6+x^5+x^4+x^3+x.$$

into two polys with coeff-sum 6 is either

- $(1,2,3,4,5,6)$ and $(1,2,3,4,5,6)$ (standard).

Any Other Non-Standard Labelings?

No.
Every way to factor

$$x(x+1)(x^2-x+1)^2(x^2+x+1) = x^8 + x^6 + x^5 + x^4 + x^3 + x.$$

into two polys with coeff-sum 6 is either

- $(1, 2, 3, 4, 5, 6)$ and $(1, 2, 3, 4, 5, 6)$ (standard).
- $(1, 2, 2, 3, 3, 4)$ and $(1, 3, 4, 5, 6, 8)$.

What About Two d-Sided Dice?

For which $d \geq 2$ are there two non-standard d-sided dice that have the same prob as standard dice? **VOTE:**

What About Two d-Sided Dice?

For which $d \geq 2$ are there two non-standard d-sided dice that have the same prob as standard dice? **VOTE:**

1. All even d.

What About Two d-Sided Dice?

For which $d \geq 2$ are there two non-standard d-sided dice that have the same prob as standard dice? **VOTE:**

1. All even d.
2. All non-prime d

What About Two d-Sided Dice?

For which $d \geq 2$ are there two non-standard d-sided dice that have the same prob as standard dice? **VOTE:**

1. All even d.
2. All non-prime d
3. Something Else

What About Two d-Sided Dice?

For which $d \geq 2$ are there two non-standard d-sided dice that have the same prob as standard dice? **VOTE:**

1. All even d.
2. All non-prime d
3. Something Else
4. UNKNOWN TO SCIENCE!

What About Two d-Sided Dice?

For which $d \geq 2$ are there two non-standard d-sided dice that have the same prob as standard dice? **VOTE:**

1. All even d.
2. All non-prime d
3. Something Else
4. UNKNOWN TO SCIENCE!

Answer on Next Slide

What About Two d-Sided Dice?

Answer There are two non-standard d-sided dice iff d is non-prime.

What About Two d-Sided Dice?

Answer There are two non-standard d-sided dice iff d is non-prime.

The proof is similar to what we did, though requires some thought.

What About d_1, d_2-Sided Dice?

For which $d_1, d_2 \geq 2$ are there non-standard d_1-sided die and d_2-sided die that have the same prob as standard dice?
VOTE:

What About d_1, d_2-Sided Dice?

For which $d_1, d_2 \geq 2$ are there non-standard d_1-sided die and d_2-sided die that have the same prob as standard dice?

VOTE:

1. One of d_1, d_2 has to be non-prime.

What About d_1, d_2-Sided Dice?

For which $d_1, d_2 \geq 2$ are there non-standard d_1-sided die and d_2-sided die that have the same prob as standard dice?
VOTE:

1. One of d_1, d_2 has to be non-prime.
2. Both d_1, d_2 have to be non-prime.

What About d_1, d_2-Sided Dice?

For which $d_1, d_2 \geq 2$ are there non-standard d_1-sided die and d_2-sided die that have the same prob as standard dice?

VOTE:

1. One of d_1, d_2 has to be non-prime.
2. Both d_1, d_2 have to be non-prime.
3. Something Else

What About d_1, d_2-Sided Dice?

For which $d_1, d_2 \geq 2$ are there non-standard d_1-sided die and d_2-sided die that have the same prob as standard dice?

VOTE:

1. One of d_1, d_2 has to be non-prime.
2. Both d_1, d_2 have to be non-prime.
3. Something Else
4. UNKNOWN TO SCIENCE!

What About d_1, d_2-Sided Dice?

For which $d_1, d_2 \geq 2$ are there non-standard d_1-sided die and d_2-sided die that have the same prob as standard dice?

VOTE:

1. One of d_1, d_2 has to be non-prime.
2. Both d_1, d_2 have to be non-prime.
3. Something Else
4. UNKNOWN TO SCIENCE!

Answer on Next Slide.

Unknown to ...

For which $d_1, d_2 \geq 2$ are there non-standard d_1-sided die and d_2-sided die that have the same prob as standard dice?

Unknown to ...

For which $d_1, d_2 \geq 2$ are there non-standard d_1-sided die and d_2-sided die that have the same prob as standard dice?

Unknown to Science

More is Known

More is Known

1. George Sicherman first posed the problem and solved it in 1978.

More is Known

1. George Sicherman first posed the problem and solved it in 1978.
   The dice produced are sometimes called **Sicherman Dice**.

More is Known

1. George Sicherman first posed the problem and solved it in 1978.
   The dice produced are sometimes called **Sicherman Dice**.
   You can buy these dice on the web!

More is Known

1. George Sicherman first posed the problem and solved it in 1978.
   The dice produced are sometimes called **Sicherman Dice**.
   You can buy these dice on the web!
2. Gasarch has an exposition on this material:
   `https://www.cs.umd.edu/~gasarch/BLOGPAPERS/billdice.pdf`

More is Known

1. George Sicherman first posed the problem and solved it in 1978.
   The dice produced are sometimes called **Sicherman Dice**.
   You can buy these dice on the web!
2. Gasarch has an exposition on this material:
   `https://www.cs.umd.edu/~gasarch/BLOGPAPERS/billdice.pdf`
3. Gallian and Rusin's paper exactly characterizes when this is possible:
   `https://www.cs.umd.edu/~gasarch/BLOGPAPERS/nonstandarddice.pdf`
   The paper only looked at n d-sided dice and I do not know of a later paper. Thats why the question of d_1, d_2 is **Unknown to Science**.

More is Known

1. George Sicherman first posed the problem and solved it in 1978.
   The dice produced are sometimes called **Sicherman Dice**.
   You can buy these dice on the web!
2. Gasarch has an exposition on this material:
   `https://www.cs.umd.edu/~gasarch/BLOGPAPERS/billdice.pdf`
3. Gallian and Rusin's paper exactly characterizes when this is possible:
   `https://www.cs.umd.edu/~gasarch/BLOGPAPERS/nonstandarddice.pdf`
   The paper only looked at n d-sided dice and I do not know of a later paper. Thats why the question of d_1, d_2 is **Unknown to Science**.
   Or maybe just **Unknown to Bill**.

Parting Thoughts

William Gasarch - University of MD

Parting Thoughts

Parting Thoughts

1. Easy to state problems about dice lead to math of interest.

Parting Thoughts

1. Easy to state problems about dice lead to math of interest.
2. Polynomials are useful for problems with dice since multiplication gives information.

Parting Thoughts

1. Easy to state problems about dice lead to math of interest.
2. Polynomials are useful for problems with dice since multiplication gives information.
3. It is remarkable that a problem about dice lead to looking at complex roots of polynomials!