HW 9 CMSC 389. DUE Jan 17
SOLUTIONS

1. (0 points) What is your name? Write it clearly. Staple your HW. When is the final? Are you free then? (if not then SEE ME IMMEDIATELY)

2. (100 points) Find a function f such that the following is true, and prove it.

   For all n there exists $f(n)$ such that for all THREE colorings of the edges of $K_{f(n)}$ there exists a homogenous set of size n

   (NOTE- in class we did TWO-coloring with $f(n) = 2^{2n}$. In the notes is TWO coloring with a slightly better bound of $f(n) = 2^{2n-2}$.)

   SOLUTION TO PROBLEM 2.

   I will FIRST DO IT the way a mathematician would do it BEFORE knowing the answer. I will then do it over again KNOWING the answer.

   NOT KNOWING. THIS IS HOW I MIGHT PRESENT IT TO MOTIVATE. YOU DO NOT NEED ALL OF THIS IN YOUR ANSWER, IT IS MORE WHAT YOU SHOULD BE THINKING.

   We need to find a function $f(n)$ such that for all 3-colorings of $K_{f(n)}$ there is a homog set of size n. Let $M = f(n)$ for now.

   Let COL be a 3-coloring of the edges of K_M. Looking ahead we will want a sequence of vertices,

   $$x_1, x_2, \ldots, x_{3n},$$

   and a sequence of colors

   $$c_1, c_2, \ldots, c_{3n},$$

   so that x_i and any vertex to its right is colored c_i. We think of x_i as being colored c_i.

   We want $3n$ of them so that we are guaranteed that there are n of the same color. (NOTE- we could get by with $3n - 2$.)

Here is the intuition: Vertex $x_1 = 1$ has M edges coming out of it. Some are RED, some are BLUE, some are GREEN. Hence there are at EITHER at least $M/3$ RED edges coming out of x_1, or at least $M/3$ BLUE edges coming out of x_1, or at least $M/3$ GREEN edges coming out of x_1. Say it was RED. Then we let $c_1 = RED$ and only keep all of the vertices v such that $COL(x_1, v) = RED$. (Or, as I say in class KILL ALL THOSE WHO DISAGREE). Then let x_2 be the least vertex left. Repeat the process with x_2.

We now describe it formally.

$$\begin{aligned} V_0 = & \quad \{1, 2, \ldots, M\} \\ x_1 = & \quad 1 \end{aligned}$$

$$c_1 = \begin{cases} \text{RED} & \text{if } |\{v \in V_0 \mid COL(\{v, x_1\}) = \text{RED}\}| \geq M/3\} \\ \text{BLUE} & \text{if } |\{v \in V_0 \mid COL(\{v, x_1\}) = \text{BLUE}\}| \geq M/3\} \\ \text{GREEN} & \text{if } |\{v \in V_0 \mid COL(\{v, x_1\}) = \text{GREEN}\}| \geq M/3\} \end{cases} \tag{1}$$

$$V_1 = \quad \{v \in V_0 \mid COL(\{v, x_1\}) = c_1\} \text{ (note that } |V_1| \geq M/3)$$

Let $i \geq 2$, and assume that V_{i-1} is defined. We define x_i, c_i, and V_i: KEY: $|V_0| = M$ and $|V_1| \geq M/3 = M/3^1$

KEY: We can assume $|V_{i-1}| \geq M/3^{i-1}$. So when we look at splitting V_i into three sets at least one of them has to be of size $M/3^i$.

$$x_i = \quad \text{the least number in } V_{i-1}$$

$$c_i = \begin{cases} \text{RED} & \text{if } |\{v \in V_{i-1} \mid COL(\{v, x_1\}) = \text{RED}\}| \geq M/3\} \\ \text{BLUE} & \text{if } |\{v \in V_{i-1} \mid COL(\{v, x_1\}) = \text{BLUE}\}| \geq M/3\} \\ \text{GREEN} & \text{if } |\{v \in V_{i-1}0 \mid COL(\{v, x_1\}) = \text{GREEN}\}| \geq M/3\} \end{cases} \tag{2}$$

$$V_i = \{v \in V_{i-1} \mid COL(\{v, x_i\}) = c_i\} \text{ (note that } |V_i| \geq M/3^i)$$

How long can this sequence go on for? Well, x_i can be defined if V_{i-1} is nonempty. Note that $|V_i| \geq M/3^i$ Since we want to go on for $3n$ iterations we take $M = 3^{3n}$.

Hence we have

$$x_1, \ldots, x_{3n}$$

Consider the colors

$$c_1, c_2, \ldots, c_{3n}$$

Each of these is either RED or BLUE or GREEN.. Hence there must be at least n of them that are the same color. Let $i_1, \ldots, i_n$ be such that $i_1 < \cdots < i_n$ and

$$c_{i_1} = c_{i_2} = \cdots = c_{i_n}$$

Denote this color by c, and consider the n vertices

$$x_{i_1}, x_{i_2}, \cdots, x_{i_n}.$$

All of the colors between then are c so we are DONE!

NOTE: We took $f(n) = 3^{3n}$.

NOW WE DO IT KNOWING THE ANSWER. THIS IS WHAT I WOULD WANT TO SEE ON A HW OR EXAM.

$$\begin{aligned} V_0 = &\ \{1, 2, \ldots, 3^{3n}\} \\ x_1 = &\ 1 \end{aligned}$$

$$c_1 = \begin{cases} \text{RED} & \text{if } |\{v \in V_0 \mid COL(\{v, x_1\}) = \text{RED}\}| \geq 3^{3n-1}\} \\ \text{BLUE} & \text{if } |\{v \in V_0 \mid COL(\{v, x_1\}) = \text{BLUE}\}| \geq 3^{3n-1}\} \\ \text{GREEN} & \text{if } |\{v \in V_0 \mid COL(\{v, x_1\}) = \text{GREEN}\}| \geq 3^{3n-1}\} \end{cases} \tag{3}$$

NOTE- at least one of the three above must happen.

$$V_1 = \{v \in V_0 \mid COL(\{v, x_1\}) = c_1\} \text{ (note that } |V_1| \geq 3^{3n-1})$$

Let $i \geq 2$, and assume that V_{i-1} is defined and $|V_{i-1}| \geq 3^{3n-i-1}$. We define x_i, c_i, and V_i:

$$x_i = \text{ the least number in } V_{i-1}$$

$$c_i = \begin{cases} \text{RED} & \text{if } |\{v \in V_{i-1} \mid COL(\{v, x_1\}) = \text{RED}\}| \geq 3^{3n-i}\} \\ \text{BLUE} & \text{if } |\{v \in V_{i-1} \mid COL(\{v, x_1\}) = \text{BLUE}\}| \geq 3^{3n-i}\} \\ \text{GREEN} & \text{if } |\{v \in V_{i-1}0 \mid COL(\{v, x_1\}) = \text{GREEN}\}| \geq 3^{3n-i}\} \end{cases} \tag{4}$$

$$V_i = \{v \in V_{i-1} \mid COL(\{v, x_i\}) = c_i\} \text{ (note that } |V_i| \geq 3^{3n-i})$$

This can go on for $3n$ iterations. Hence we have

$$x_1, \ldots, x_{3n}$$

Consider the colors

$$c_1, c_2, \ldots, c_{3n}$$

Each of these is either RED or BLUE or GREEN.. Hence there must be at least n of them that are the same color. Let $i_1, \ldots, i_n$ be such that $i_1 < \cdots < i_n$ and

$$c_{i_1} = c_{i_2} = \cdots = c_{i_n}$$

Denote this color by c, and consider the n vertices

$$x_{i_1}, x_{i_2}, \cdots, x_{i_n}.$$

All of the colors between then are c so we are DONE!

THINK ABOUT: What if we had 4 colors? 5 colors? c colors?

3. READ the paper on APPLICATION OF RAMSEY THEORY TO HISTORY on the course website. There will be a SHORT QUIZ on it at the beginning of the next lecture. (NOTE- its an easy read.)