An Application of Ramsey's Theorem to Proving Programs Terminate: An Exposition

William Gasarch-U of MD

Ramsey Kings Paper—Which Names Are Fake?- Part I

1. Eugene Wigner- REAL. I'm surprised to!
2. Herbert Scarf- REAL
3. Samuel Harrington- REAL
4. Dorwin Cartrwright-REAL. Really? Yes!
5. Frank Harary-REAL
6. Charles Percy Snow-REAL
7. Jacob Fox-REAL, as his his sociology story.
8. Sandor Szalai-REAL. You've got to be kidding.
9. Paul Erdos-REAL. I think you knew that.
10. Paul Turan-REAL.
11. Vera Sos-REAL.

Which Names Are Fake?- Part II

The names are all anagrams of real people that are similar.

1. Sir Woodson Kneading–Anagram is Doris Kearns Goodwin.
2. H.K. Donnut–Anagram is Don Knuth.
3. Moss Chill Beaches–Anagram is Michael Beschloss.
4. Tim Andrer Grant–Anagram is Martin Gardner.
5. Alma Rho Grand–Anagram is Ronald Graham
6. D.H.J. Polymath–see polymath project.
7. Ana Writset–Anagram is Ian Stewart.
8. Tee A. Cornet–Anagram is Terence Tao.
9. Andy Parrish-REAL.
10. Stephen Fenner-REAL.
11. Clyde Kruskal-REAL on a good day.

Ramsey Theory and PL (Serious)

1. Work by
   1.1 Floyd,
   1.2 Byron Cook, Andreas Podelski, Andrey Rybalchenko,
   1.3 Lee, Jones, Ben-Amram
   1.4 Others
2. Pre-Apology: Not my area-some things may be wrong.
3. Pre-Brag: Not my area-some things may be understandable.

Overview I

Problem: Given a program we want to prove it terminates no matter what user does (called TERM problem).

1. Impossible in general- Harder than Halting.
2. But can do this on some simple progs. (We will.)
3. Some of the proof use Ramsey Theory!

Notation

1. Will use psuedo-code progs.
2. KEY: If A is a set then the command

   ```
   x = input(A)
   ```

   means that x gets some value from A that the user decides.
3. Note: we will want to show that no matter what the user does the program will halt.
4. The code

   ```
   (x,y) = (f(x,y),g(x,y))
   ```

   means that simultaneously x gets f(x,y) and y gets g(x,y).

Easy Example of Traditional Method

```
(x,y,z) = (input(INT), input(INT), input(INT))
While x>0 and y>0 and z>0
        control = input(1,2,3)
        if control == 1 then
                (x,y,z)=(x+1,y-1,z-1)
        else
        if control == 2 then
                (x,y,z)=(x-1,y+1,z-1)
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```

Sketch of Proof of termination:

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Whatever the user does x+y+z is decreasing.

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Sketch of Proof of termination:

Whatever the user does x+y+z is decreasing.

Eventually x+y+z=0 so prog terminates there or earlier.

What is Traditional Method?

General method due to Floyd: Find a function f(x,y,z) from the values of the variables to N such that

1. in every iteration f(x,y,z) decreases
2. if f(x,y,z) is every 0 then the program must have halted.

Note: Method is more general- can map to a well founded order such that in every iteration f(x,y,z) decreases in that order, and if f(x,y,z) is ever a min element then program must have halted.

Hard Example of Traditional Method

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Sketch of Proof of termination:
Use Lex Order: $(0,0,0) < (0,0,1) < \cdots < (0,1,0) \cdots$.
Note: $(4, 10^{100}, 10^{10!}) < (5,0,0)$.

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In every iteration (x,y,z) decreases in this ordering.
If hits bottom then all vars are 0 so must halt then or earlier.

Notes about Proof

1. Bad News: We had to use a funky ordering. This might be hard for a proof checker to find. (Funky is not a formal term.)
2. Good News: We only had to reason about what happens in one iteration.

Keep these in mind- our later proof will use a nice ordering but will need to reason about a block of instructions.

Recall RamseyTheory for Infinite Graphs

Definition

Let $c, k, n \in \mathbb{N}$. K_n is the complete graph on n vertices (all pairs are edges). K_ω is the infinite complete graph. A c-coloring of K_n is a c-coloring of the edges of K_n. A homogeneous set is a subset H of the vertices such that every pair has the same color (e.g., 10 people all of whom know each other).

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2. For all c-colorings of $K_{c^{ck-c}}$ there is a homog k-set.
3. For all c-colorings of the K_ω there exists a homog ω-set.

Alt Proof Using Ramsey

```
(x,y,z) = (input(INT),input(INT),input(INT))
While x>0 and y>0 and z>0
        control = input(1,2)
        if control == 1 then
                (x,y) =(x-1,input(y+1,y+2,...))
        else
                (y,z)=(y-1,input(z+1,z+2,...))
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Begin Proof of termination:

Alt Proof Using Ramsey

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Begin Proof of termination:
If program does not halt then there is infinite sequence
$(x_1, y_1, z_1), (x_2, y_2, z_2), \ldots,$ representing state of vars.

```
control = input(1,2)
if control == 1 then
        (x,y) =(x-1,input(y+1,y+2,...))
else
        (y,z)=(y-1,input(z+1,z+2,...))
```

Reasoning about Blocks

```
control = input(1,2)
if control == 1 then
        (x,y) =(x-1,input(y+1,y+2,...))
else
        (y,z)=(y-1,input(z+1,z+2,...))
```

Look at $(x_i, y_i, z_i), \ldots, (x_j, y_j, z_j)$.

1. If control is ever 1 then $x_i > x_j$.
2. If control is never 1 then $y_i > y_j$.

Reasoning about Blocks

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control = input(1,2)
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Look at $(x_i, y_i, z_i), \ldots, (x_j, y_j, z_j)$.

1. If control is ever 1 then $x_i > x_j$.
2. If control is never 1 then $y_i > y_j$.

Upshot: For all $i < j$ either $x_i > x_j$ or $y_i > y_j$.

Use Ramsey

If program does not halt then there is infinite sequence $(x_1, y_1, z_1), (x_2, y_2, z_2), \ldots,$ representing state of vars.
For all $i < j$ either $x_i > x_j$ or $y_i > y_j$.
Define a 2-coloring of the edges of K_ω:

$$COL(i,j) = \begin{cases} X \text{ if } x_i > x_j \\ Y \text{ if } y_i > y_j \end{cases} \quad (1)$$

By Ramsey there exists homog set $i_1 < i_2 < i_3 < \cdots$.
If color is X then $x_{i_1} > x_{i_2} > x_{i_3} > \cdots$
If color is Y then $y_{i_1} > y_{i_2} > y_{i_3} > \cdots$
In either case will have eventually have a var ≤ 0 and hence program must terminate. Contradiction.

Compare and Contrast

1. Trad. proof used lex order on N^3–complicated!
2. Ramsey Proof used only used the ordering N.
3. Traditional proof only had to reason about single steps.
4. Ramsey Proof had to reason about blocks of steps.

What do YOU think?

VOTE:

1. Traditional Proof!
2. Ramsey Proof!
3. Stewart/Colbert in 2016!

Didn't Need Full Strength of Ramsey

The colorings we applied Ramsey to were of a certain type:

Definition
A coloring of the edges of K_n or K_{N} is transitive if, for every $i < j < k$, if $COL(i,j) = COL(j,k)$ then both equal $COL(i,k)$.

1. Our colorings were transitive.
2. Transitive Ramsey Thm is weaker than Ramsey's Thm.

Transitive Ramsey Weaker than Ramsey

TR is Transitive Ramsey, R is Ramsey.

$$R(k,c) \sim c^{ck}$$

But

$$TR(k,c) \sim k^{c}$$

Summary

1. Ramsey Theory can be used to prove some simple programs terminate that seem harder to do my traditional methods. Interest to PL and to YOU!
2. Full strength of Ramsey not needed. Interest to Logicians and Combinatorists and Douglas and YOU!