# The Hamiltonian Cycle Problem is NP-Complete

Karthik Gopalan

**CMSC 452** 

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4 E b November 25, 2014 1 / 31

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# Section 1

Outline

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November 25, 2014 2 / 31

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#### 2 3-SAT $\leq_P$ Directed Ham Path

- Procedure
- Construction
- Examples
- A Dialog

#### 3 Hamiltonian Path ≤<sub>P</sub> Hamiltonian Cycle

#### 4 3-SAT $\leq_P$ Undirected Planar Hamiltonian Cycle

- Gadgets
- Construction

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# Section 2

# Introduction

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Introduction

#### What is a Hamiltonian Cycle

A cycle through a graph G = (V, E) that touches every vertex once.



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Introduction

### Hamiltonian Path $\in NP$

**()** The certificate: a path represented by an ordering of the verticies

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Introduction

# Hamiltonian Path $\in NP$

- The certificate: a path represented by an ordering of the verticiesVerify:
  - - Each node is in the path once
    - An edge exists between each consecutive pair of nodes

3-SAT  $\leq_P$  Directed Ham Path

# Section 3

# 3-SAT $\leq_P$ Directed Ham Path

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#### Subsection 1

Procedure

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#### Procedure

#### Start with a 3-CNF formula

$$\phi = (a_1 \lor b_1 \lor c_1) \land (a_2 \lor b_2 \lor c_2) \land \dots \land (a_k \lor b_k \lor c_k)$$

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## Procedure

#### Start with a 3-CNF formula

$$\phi = (\mathsf{a}_1 \lor \mathsf{b}_1 \lor \mathsf{c}_1) \land (\mathsf{a}_2 \lor \mathsf{b}_2 \lor \mathsf{c}_2) \land \dots \land (\mathsf{a}_k \lor \mathsf{b}_k \lor \mathsf{c}_k)$$

**②** Create a graph G that has a Hamiltonian Path iff  $\phi$  is satisfiable

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#### Subsection 2

#### Construction

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Represent each variable  $x_i$  with a gadget with 3k + 3 nodes:



# 2 Ways



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# 2 Ways



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#### Clauses

Each clause  $c_i$  is represented as a single node:



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#### Connect the Variables



#### Connect the Clauses



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#### Connect the Clauses



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15 / 31

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#### Subsection 3

#### Examples

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16 / 31

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 $(x \lor y) \land (\overline{x} \lor \overline{y})$ 



 $x \wedge \overline{x}$ 



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#### Subsection 4

A Dialog

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19 / 31

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• Gasarch [to class]: So, could you code this up?

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- Gasarch [to class]: So, could you code this up?
- Class: Yes!

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- Gasarch [to class]: So, could you code this up?
- Class: Yes!
- Gasarch: Would you want to?

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- Gasarch [to class]: So, could you code this up?
- Class: Yes!
- Gasarch: Would you want to?
- Class: No...

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- Gasarch [to class]: So, could you code this up?
- Class: Yes!
- Gasarch: Would you want to?
- Class: No...
- Scott: Maybe in Prolog.

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- Gasarch [to class]: So, could you code this up?
- Class: Yes!
- Gasarch: Would you want to?
- Class: No...
- Scott: Maybe in Prolog.
- Liz: If you paid me.

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- Gasarch [to class]: So, could you code this up?
- Class: Yes!
- Gasarch: Would you want to?
- Class: No...
- Scott: Maybe in Prolog.
- Liz: If you paid me.
- Gasarch: What happened to the love of computer science! When Clyde was an undergrad... The answer is 0 or on the board!... blah blah blah

Hamiltonian Path  $\leq_P$  Hamiltonian Cycle

## Section 4

# Hamiltonian Path $\leq_P$ Hamiltonian Cycle

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Hamiltonian Path  $\leq_P$  Hamiltonian Cycle

# Question 4 on the Final

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22 / 31

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# Section 5

# 3-SAT $\leq_P$ Undirected Planar Hamiltonian Cycle

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#### Subsection 1

Gadgets

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#### Or



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#### Or



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## X-Or



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26 / 31

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## X-Or



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#### Subsection 2

#### Construction

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28 / 31

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#### Clauses



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3-SAT  $\leq_P$  Undirected Planar Hamiltonian Cycle Construction

 $(x \lor y \lor z) \land (\overline{x} \lor \overline{y} \lor w) \land (y \lor \overline{z} \lor \overline{w})$ 



3-SAT  $\leq_P$  Undirected Planar Hamiltonian Cycle Construction

#### That's BS Man!



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