452 FINAL

Do not open this exam until you are told. Read these instructions:

1. This is a closed book exam, though ONE sheet of notes is allowed. **No calculators, or other aids are allowed**. If you have a question during the exam, please raise your hand.
2. There are 6 problems which add up to 100 points. The exam is 2 hours.
3. For each question show all of your work and **write legibly**. **Clearly indicate** your answers. No credit for illegible answers.
4. After the last page there is paper for scratch work. If you need extra scratch paper **after** you have filled these areas up, please raise your hand. Scratch paper must be turned in with your exam, with your name and ID number written on it, but scratch paper **will not** be graded.
5. Please write out the following statement: "*I pledge on my honor that I will not give or receive any unauthorized assistance on this examination.*"

6. Fill in the following:

NAME :
SIGNATURE :
SID :
SECTION NUMBER :

SCORES ON PROBLEMS

Prob 1:	
Prob 2:	
Prob 3:	
Prob 4:	
Prob 5:	
Prob 6:	
TOTAL	

1. (10 points) (NOTE- WE DID NOT DO THE EXTENDED PUMPING LEMMA OR COMM COMPLEXITY. EVEN SO, YOU SHOULD BE ABLE TO PROVE THIS LANG NOT REG USING OUR PUMPING LEMMA.) Prove that the following language is NOT regular. You may use EITHER the extended pumping lemma (what I've called the Aaron George PL), OR communication complexity.

$$\{a^n b^{n^2} : n \in \mathsf{N}\}.$$

SOLUTION TO PROBLEM ONE

Omitted.

2. (20 points- 5 points each.) Give an example of each of the following. No proof requires but be clear as to what your language is.

   (a) A language that is context free but not regular.

   (b) A language that is in P but not context free

   (c) A language that is NP-complete that DOES NOT have to do with graphs and DOES NOT have to do with boolean formulas.

   (d) A language that is undecidable.

   SOLUTION TO PROBLEM TWO

   2a) $a^n b^n$

   2b) $a^n b^n c^n$

   2c) $\{(A, b) : Ax \leq b$ has a 0-1 solution $\}$

   2d) HALT

3. (20 points) For this problem you can assume the following theorem: *if x, y are relatively prime and $n = xy - x - y$ then (1) n CANNOT be written as x's and y's, but (2) EVERY $i \geq n+1$ CAN be so written.*

   Let
$$L = \{a^i : i \neq 58\}$$

   Describe an NFA for L which has ≤ 35 states. PROVE that it works.

   SOLUTION TO PROBLEM THREE

   Let $x = 8$ and $y = 9$. Then $n = 8 * 9 - 8 - 9 = 72 - 17 = 55$.

   The NFA has

   (a) One branch that has a chain of length 3 and a loop of size 9 and a shortcut so that it can go just 8. This branch will accept $\{a^i : i \geq 59\}$ and some other stuff but it will NOT accept a^{58}. This branch has $3 + 9 = 12$ states.

   (b) One branch that accepts all $\{a^i : i \equiv 1 \pmod 2\}$. This branch has two states. (NOTE- this branch DOES NOT accept a^{58}.)

   (c) One branch that accepts all $\{a^i : i \equiv 0, 2 \pmod 3\}$. This branch has three states. (NOTE- this branch DOES NOT accept a^{58}.)

   (d) One branch that accepts all $\{a^i : i \equiv 0, 1, 2, 4 \pmod 5\}$. This branch has five states. (NOTE- this branch DOES NOT accept a^{58}.)

   (e) One branch that accepts all $\{a^i : i \equiv 0, 1, 3, 4, 5, 6 \pmod 7\}$. This branch has seven states. (NOTE- this branch DOES NOT accept a^{58}.)

   If a^i is REJECTED then

   (a) Because of the big-loops-branch $i \leq 58$.

   (b) Because of the mod-2 branch $i \equiv 0 \pmod 2$.

   (c) Because of the mod-3 branch $i \equiv 1 \pmod 3$.

   (d) Because of the mod-5 branch $i \equiv 3 \pmod 5$.

   (e) Because of the mod-7 branch $i \equiv 2 \pmod 7$.

Since $i \equiv 0 \pmod 2$ and $i \equiv 1 \pmod 3$, $i \equiv 4 \pmod 6$

Since $i \equiv 3 \pmod 5$ and $i \equiv 2 \pmod 7$, $i \equiv 23 \pmod{35}$

Since $i \equiv 4 \pmod 6$ and $i \equiv 23 \pmod{35}$, $i \equiv 58 \pmod{210}$

The only such i that is ≤ 59 IS 58. So a^{58} is the only rejected string.

4. (40 points- 5 points each)

   Throughout this problem you may assume $P \neq NP$. For EACH of the sets ON THE NEXT PAGE say either (1) say that its IN P and give an algorithm for it, OR (2) say that its NP-Complete and give a reduction for it. You may assume the following problems are KNOWN to be NP-complete

$$IS = \{(G, k) : \ G \text{ has an Ind Set of size } k \ \}$$

   (An Ind Set is a set of vertices that all DO NOT know each other.)

$$IS2 = \{G : G \text{ is a graph on } n \text{ vertices that has an Independent Set of size } \lfloor n/3 \rfloor \ \}$$

$$VC = \{(G, k) : G \ G \text{ has a Vertex Cover of size } k \ \}$$

   (A Vertex Cover is a set of vertices such that every edge has at least one of those vertices as an end point.)

$$COL = \{(G, k) : G \text{ is } k\text{-colorable } \}$$

$$3\text{-}SAT = \{\phi : \ \phi \text{ is in 3-CNF form and there is a satisfying assignment for it } \}$$

   (3-CNF is $C_1 \wedge C_2 \wedge \cdots \wedge C_\ell$ where each C_i is an OR of THREE literals. The C_i's are called CLAUSES.)

(a)

$$\{G : G \text{ is a graph on } n \text{ vertices that has an Independent Set of size} \geq 17\ \}$$

(b)

$$\{G : G \text{ is a graph on } n \text{ vertices that has an Independent Set of size} \geq \lfloor n/2 \rfloor\ \}$$

(c)

$$\{G : G \text{ is a graph on } n \text{ vertices that has a Vertex Cover of size} \geq 17\ \}$$

(d)

$$\{G : G \text{ is a graph on } n \text{ vertices that has a Vertex Cover of size} \geq \lfloor \log_2 n \rfloor\ \}$$

(e)

$$\{G : G \text{ is a graph on } n \text{ vertices that is 17-colorable}\ \}$$

(f)

$$\{\phi :\ \phi \text{ is in 3-CNF form and there is a way to satisfy} \geq 17 \text{ of the clauses}\ \}$$

(g)

$$\{\phi :\ \phi \text{ is in DNF form and } \phi \in SAT\ \}$$

(DNF is $C_1 \vee C_2 \vee \cdots \vee C_\ell$ where each C_i is an AND of literals.)

(h)

$$\{M :\ M \text{ is an NFA such that SOME strings is accepted}\ \}$$

SOLUTION TO PROBLEM FOUR

4a) This is in P. Try all possible subsets of size 17, takes $O(n^{17})$ steps.

4b) This is NPC. Given G add m vertices- we will determine m soon

G has n vertices and and IS of size $n/3$ implies

$G \cup H$ has $n + m$ vertices and an IS of size $n/3 + m$.

Need $n/3 + m = (n + m)/2$

$n/3 + m = n/2 + m/2$

$m/2 = n/6$

$m = n/3$

SO add $n/3$ vertices.

4c) P. Either brute force all 17 or be clever.

4d) P. Need to be clever- tree is of size $2^{\log n}$ = n.

4e) NPC. G maps to G union K_{14} with every edge between the two.

4f) P. Look at all sets of 17 clauses.

4g) P. DNF form- all we need is for ONE of the parts to be true.

Algorithm: If there is SOME part with no literals and their negations then output YES. If every part has some x such that both x and $\neg x$ are in it, then NO.

4h) P.

A_0 is the set containing the start state

A_{i+1} is A_i union all the states you can get from it.

If $A_n \cap F \neq \emptyset$ then YES, otherwise no.

5. (10 points) Let

$$L = \{G : G \text{ is a graph on } n \text{ vertices that is } n-17\text{-colorable } \}$$

Either show $L \in P$ or show L is NP-complete.

SOLUTION TO PROBLEM SIX

In P. For every set of 17 vertices take its COMPLEMENT and color it with all different colors say $\{1, \ldots, n-17\}$. Then try to color the 17 vertices left with the $n-17$ colors in all ways. This takes $(n-17)^{17}$ steps.

Scratch Paper