452 FINAL

<u>Do not open this exam until you are told.</u> Read these instructions:

1. This is a closed book exam, though ONE sheet of notes is allowed. **No calculators, or other aids are allowed**. If you have a question during the exam, please raise your hand.
2. There are 5 problems which add up to 100 points. The exam is 2 hours.
3. For each question show all of your work and **write legibly**. **Clearly indicate** your answers. No credit will be given for illegible answers.
4. Please write out the following statement: "*I pledge on my honor that I will not give or receive any unauthorized assistance on this examination.*"

Fill in the following:

NAME :
SIGNATURE :
SID :
SECTION NUMBER :

1. (20 points- 4 points each)

   For this problem: (1) assume $P \neq NP$, (2) let $M_1, M_2, \ldots$ be a standard list of Turing Machines, (3) assume Planarity is in P. For each of the following languages L say which of the following is true.

   - L is REGULAR (just write REG).
   - L is in P but not REGULAR (just write P).
   - L is in NP but not P (just write NP).
   - L is Decidable but not in NP (just write DEC).
   - L is in Σ_1 but not decidable (just write SIGMAONE).
   - L is NOT in Σ_1 (just write NOT SIGMAONE).

   NO PROOF REQUIRED! +4 for being right, -3 for being wrong. (But you can't get below 0.)

   (a) $\{a^{2^n} \mid n \geq 100\}$
   PUT ANSWER HERE:

   (b) In WS1S notation: $\{(x, y, z) \mid x + y = z\}$
   PUT ANSWER HERE:

   (c) $\{G \mid \ G$ has a clique of size 6 $\}$
   PUT ANSWER HERE:

   (d) $\{G \mid \ G$ can be drawn in the plane with ≤ 84 crossings $\}$
   PUT ANSWER HERE:

   (e) $\{x \in \{0,1\}^* \mid C(x) \leq 2|x|\}$ ($C(x)$ is the Kolmogorov complexity of x, $|x|$ is the length of x.)
   PUT ANSWER HERE:

SOLUTION TO PROBLEM ONE

(a) through (d) are all in P, (e) is REG.

Proofs of (a) through (d) I leave to the reader

For (e) note that all but a finite number of strings will have $C(x) \leq 2|x|$ by the program

PRINT(x)

So (e) is a set whose complement is finite, so whose complement is regular, so its regular.

2. (20 points- 10 points each) For this problem you may USE the following TWO theorems:

Theorem ONE: If x, y are relatively prime then

- For all $z \geq xy - x - y + 1$ there exists $c, d \in \mathsf{N}$ such that $z = cx + dy$.
- There is no $c, d \in \mathsf{N}$ such that $xy - x - y = cx + dy$.

Theorem TWO: Let $n \in \mathsf{N}$, and let

$$L_n = \{a^i \mid i \neq n\}.$$

Any DFA for L_n REQUIRES at least n states.

And NOW on to the problem.

Start your answeron the next page (the back of this one) and can also use the page after that.

(a) Let

$$L = \{a^i \mid i \neq 185\}.$$

Assume $\Sigma = \{a\}$. Does there exist an NFA for L with less than 50 states? If so then draw the NFA; you may use DOT DOT DOT. (You DO NOT have to prove that it works.) If not then PROVE there is no such NFA.

(b) Give a function f such that the following hold:

- $\lim_{n \to \infty} f(n) = \infty$,
- For all $n \geq 1000$ there DOES NOT EXIST an NFA for

$$L = \{a^i \mid i \neq n\}$$

with $\leq f(n)$ states in it.

You need to PROVE your result (recall that you can use the Theorems ONE or TWO or both).

SOLUTION TO PROBLEM TWO-A

Solution to part a is standard so just sketch. Take $x = 14$ and $y = 15$ so

$$15 \times 14 - 15 - 14 = 181$$

So need loop of size 15, tail of size $185 - 181 = 4$, primes loops of sizes $2, 3, 5, 7$, and an extra state to connect everything. So number of states is

$$15 + 1 + 4 + 2 + 3 + 5 + 7 = 37$$

Comment on grading: I am amazed anyone got this wrong, but those that did were so far off there is nothing to comment on .

SOLUTION TO PROBLEM TWO-B

Solution to part b: $f(n) = \lfloor \lg_2 n \rfloor - 1$ since if there was such an NFA then there would be a DFA with $\le n - 1$ states.

Comment on grading:

- Any answer that was increasing and $\le \lg_2(n)$ and had the DFA-explanation was full credit- 10 points.
- If a log-answer was given but the explanation was bad then 0 points.
- Any answer LARGER than *logn* got 0 points. Some people had answers like $\sqrt(n)$ and tried to say that since THEY could not do it using THEOREM ONE with any better than $\sqrt(n)$ states then it could not be done! WOW! I didn't know we had such brilliant people in our class so that if THEY can't do it, IT CAN"T BE DONE! Seriously, one of the points of the course (and indeed the point of the segment on small NFA's) was that just because YOU can't solve a problem a certain way DOES NOT mean it can't be solved. This course is supposed to teach you to RESPECT how hard it is to prove LOWER BOUNDS (e.g., an NFA REQUIRES

blah states) since you have to rule out EVERY TRICK POSSIBLE. So it is my hope that the people who got it wrong and are now reading this are ENLIGHTNTED that lower bounds require proving that NO TECHNIQUE will work, not just that YOUR TECHNIQUE or MY TECHNIQUE will not work.

3. (20 points- 10 points each)

   For each of the following sets WRITE IT using quantifiers and try to get it as low as possible in the arithmetic hierarchy (i.e., given set X try to find the least $n \in \mathsf{N}$ such that $X \in \Sigma_n$ or $X \in \Pi_n$). STATE for each one where it is (e.g., $X \in \Sigma_{452}$).

   (a)
   $$A = \{e \mid M_e \text{ halts on exactly 3 inputs }\}.$$

   (b)
   $$B = \{(x, a) \mid C(x) \le a\}.$$

   ($C(x)$ is the Kolmogorov complexity of x.)

   You DO NOT need to prove your answers are correct, but it should be CLEAR that they are.

SOLUTION TO PROBLEM THREE-A

$$A = \{e \mid (\exists x_1, x_2, x_3, s)(\forall x, t)$$

- All the x_i are different
- $M_{e,s}(x_1) \downarrow$ AND $M_{e,s}(x_2) \downarrow$ AND $M_{e,s}(x_3) \downarrow$.
- If $x \notin \{x_1, x_2, x_3\}$ then $M_{e,t}(x) \uparrow$.

$$A \in \Sigma_2.$$

COMMENTS ON GRADING

- Saying Σ_2 with on explanation or a bogus explanation was 0 points.
- Note that you need to have the time bound s in there. If you don't, then 5 points.

SOLUTION TO PROBLEM THREE-B

$$B = \{(x, a) \mid (\exists e, s)[M_{e,s}(0) \downarrow = x, |M_e| \le a]\}$$

$$B \in \Sigma_1.$$

COMMENTS ON GRADING

- Saying Σ_1 with on explanation or a bogus explanation was 0 points.
- Note that you need to have the time bound s in there. If you don't, then 5 points.

4. (20 points) Give a polynomial-time algorithm that decides whether a given 2-CNF formula is in 2-SAT. You can assume that every clause has *exactly* 2 literals. You DO NOT need to prove the runtime or correctness of the algorithm.

   (NOTE: Your algorithm should only determine if the formula is satisfiable or not. It should NOT output any satisfying assignment.)

   (NOTE: You may assume a poly-time algorithm for DFS is provided. Your algorithm should be clear, but not too verbose. Unclear solutions will lose points.)

SOLUTION TO PROBLEM FOUR

Omitted

COMMENTS ON GRADING

People lost points

- Lack of clarify. Losses of 10 to 20 points.
- Nonsense- such as reducing 2-SAT to an NP-complete problem. Lose all 20.
- Taking the graph and incorrectly determining if a literal is contained in a cycle along with its negation. Lost 10.
- Constructing the graph improperly to begin with. Lost 10.

5. (20 points) Let

$$TOT = \{e \mid (\forall x)(\exists s)[M_{e,s}(x) \downarrow]\}$$

(the set of all Turing machines that halt on all inputs).

Find a function $a(n)$ such that the following is true, and PROVE your result.

If f can be computed with n queries to TOT then there exists $a(n)$ partial computable functions $f_1, \ldots, f_{a(n)}$ such that, for all x,

$$f(x) \in \{f_1(x), \ldots, f_{a(n)}(x)\}.$$

(If some $f_i(x)$ does not exist then $f(x)$ cannot be that one. For example, if $f_1(x)$ and $f_2(x)$ do not exist but the rest do then

$$f(x) \in \{f_3(x), \ldots, f_{a(n)}(x)\}.)$$

SOLUTION TO PROBLEM FIVE

$a(n) = 2^n$.

For each $\sigma \in \{Y, N\}^n$ let f_σ be the TM that runs the oracle TM but when asked the ith queestion answers the ith bit of σ.

For all x SOME σ is the correct sequence of answers, so that $f_{\sigma(x)}$ will converge and be correct.

Hence, for all x

$$f(x) \in \{f_\sigma(x) \mid \sigma \in \{Y, N\}^n\}.$$

COMMENTS ON GRADING

- If you didn't have $a(n) = 2^n$ you got 0 points.
- If you had $a(n) = 2^n$ but your explanation was bogus (e.g., mentioned binary search) then you got 5.