1. (30 points) The alphabet is \{a, b\}. Let \( n \geq 0 \) and let

\[ L_n = \{a, b\}^* a \{a, b\}^n \]

(so the \((n + 1)\)th letter from the end is \(a\)).

(a) (15 points) Draw a DFA for \(L_n\) when \(n = 2\). Describe the DFA for \(L_n\) for any general \(n\). How many states does \(L_n\) have in general as a function of \(n\)?

(b) (15 points) Draw an NFA for \(L_n\) for any general \(n\). You may use DOT DOT DOT and other shortcuts. How many states does it have as a function of \(n\)?

(c) (0 points) THINK ABOUT proving that any DFA for \(L_n\) has LOTS of states.

2. (30 points) Use the conventions about representing numbers and sets established in class. Your DFA’s should have ACCEPT states (labelled A), REJECT states (labelled R), and STUPID states (labelled S).

(a) (15 points) Draw a DFA for

\[ \{(x, A) \mid x + 1 \in A\} \]

How many states does it have?

(b) (15 points) For all \(n\) draw a DFA (you may use DOT DOT DOT) for

\[ L_n = \{(x, A) \mid x + n \in A\} \]

How many states does it have as a function of \(n\)?

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3. (20 points) Note that $47 \times 101 = 4747$. Let

$$L = \{a^n \mid n \not\equiv 0 \pmod{4747}\}.$$ 

There is clearly a DFA for $L$ with 4747 states.

(a) (10 points) Prove that ANY DFA for $L$ has to have $\geq 4747$ states.

(b) (10 points) Prove or Disprove: There is an NFA for $L$ with $< 4747$ states.

**SOLUTION TO PROBLEM THREE**

a) Assume that there is a DFA for $L$ with $< 4747$ states. Input $a^{4747}$ to this DFA. In its run there must be a repeated state. Hence there exist numbers $i$ and $j$ with $1 \leq i < j \leq 4747$ such that

$a^i$ and $a^j$ both end up in state $q$.

Then $a^i a^{4747-j} = a^{4747+i-j} \not\equiv a^{4747}$ and $a^i a^{4747-j} = a^{4747}$ both end up in the same state $q$. But the first string should be rejected and the second one accepted. This is a contradiction.

b) Omitted for now- will post later. I have my reasons.

b) YES there is an NFA for $L$ with MUCH LESS than 4747 states.

Let $n \not\equiv 4747$. Note that $n$ CANNOT be BOTH $\equiv 0 \pmod{101}$ and $\equiv 0 \pmod{47}$ (if it was then it would be $\equiv 0 \pmod{4747}$). Hence either

- There exists $i \in \{1, \ldots, 46\}$, $n \equiv i \pmod{47}$, or
- There exists $i \in \{1, \ldots, 100\}$, $n \equiv i \pmod{101}$.

Hence your NFA does the following: two e-transitions from the start state: (1) one of them goes to a DFA that accepts iff $n \not\equiv 0 \pmod{47}$ (this takes 47 states), (2) the other goes to a DFA that accepts iff $n \not\equiv 0 \pmod{101}$ (this takes 101 states)

Hence there is an NFA for $L$ with 148 states plus the start state, so 149 states MUCH less than 4747.

**END OF SOLUTION TO PROBLEM THREE**
4. (20 points) Write pseudo code for an algorithm that will, GIVEN a DFA $M$ determine if $L(M) \neq \emptyset$ or not. (Here, $L(M)$ denotes the language that $M$ accepts.) It must be completely self-contained, so you can’t say something like Use Kruskal’s MST algorithm here and pay Clyde’s Uncle Joe the Royalties in the algorithm (HINT: that is not part of any correct answer that I know of anyway.)

**SOLUTION TO PROBLEM 4**

(a) Input DFA $M = (Q, \Sigma, \delta, s, F)$.
(b) $X_0 := \{s\}$
(c) $n := |Q|$
(d) For $i := 1$ to $n$ $X_i := X_{i-1} \cup \{\delta(q, \sigma) \mid q \in X_{i-1}, \sigma \in \Sigma\}$
(e) If $X_n \cap F \neq \emptyset$ then output $L(M) \neq \emptyset$.
(f) If $X_n \cap F = \emptyset$ then output $L(M) = \emptyset$. 