NFA for $\{a^y : y \neq 1000\}$
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1 Introduction

Let

$$L = \{a^y : y \neq 1000\}$$

It is easy to show that any DFA for L requires 1002 states.
We show that there is an NFA for L with *substantially fewer states*.

2 The Big Loop

Theorem:

1. For all $n \geq 992$ there exists $x, y \in N$ such that $n = 32x + 33y$.
2. There does not exist $x, y \in N$ such that $991 = 32x + 33y$.

Proof:
a) We prove this by induction on n.
Base Case: $n = 992$. $992 = 32 \times 31 + 33 \times 0$.

Ind Hyp: Assume that $n \geq 992$ and that there exists $x, y \in N$ such that $n = 32x + 33y$.

Ind Step: We have $n = 32x + 33y$. We want to get $n + 1 = 32x' + 33y'$. Thinking in terms of coins we want to either:

- Remove some 32-cent coins, add some 33-cent coins, and be up by 1. In this case you need to HAVE some 32 cent coins. This one is easy- we need to remove a 32-cent coin and add a 33-cent coin. So we'll need at least 1 32-cent coin.
- Remove some 33-cent coins, add some 32-cent coins, and be up by 1. In this case you need to HAVE some 33 cent coins. This one is a bit harder- note that $33 \times 31 = 1023$ and $32 \times 32 = 1024$. So we want to remove 31 33-cent coins, and add 32 32-cent coins. We'll need to have ≥ 31 33-cent coins.

These become two cases, and a third case to show that the first two are all that can happen.
Case 1: $x \geq 1$. Then

$$n + 1 = 32(x - 1) + 33(y + 1)$$

Case 2: $x = 0$ and $y \geq 31$. Then

$$n = 33y$$

$$n + 1 = 32 \times 32 + 33(y - 31)$$

(Note that $32^2 - 33 \times 31 = 1$ because more generally

$$z^2 - (z+1)(z-1) = 1.$$

)
Case 3: $x \le 0$ AND $y \le 30$. Then

$$n = 32x + 33y \le 32 \times 0 + 33 \times 30 = 990 < 992$$

This cannot happen. Hence Case 3 cannot occur.

b) Assume, by way of contradiction, that there exists $x, y \in N$ such that $991 = 32x + 33y$.
Take the equation mod 32

$$991 \equiv 0 \times x + 1 \times y \pmod{32}$$

$$31 \equiv y \pmod{3}2.$$

So $y \ge 31$. So

$$991 = 32x + 33y \ge 31 \times 0 + 33 \times 31 = 1023$$

This is a contradiction.
End of Proof
The following corollary of the theorem above is easy:
Corollary:

1. For all $n \ge 1001$ there exists $x, y \in N$ such that $n = 32x + 33y + 9$.
2. There does not exist $x, y \in N$ such that $1000 = 32x + 33y + 9$.

Theorem: There exist an NDFA M on 42 states such that the following is true:

1. For all $n \ge 1001$, M accepts a^n.
2. M does not accepts a^{1000}.
3. We have NO COMMENT on the behaviour of M on a^i for $i \le 999$.

Proof:
The NFA has a start state and then a string of 9 states, so that
state 0 is the start state
state 1 is the next state- you are here if input is a^1.
$\vdots$
state 9 is the next state- you are here if input is a^9.
(States 0,1,2,3,4,5,6,7,8 are reject states, though we could make the accept. State 9 is accept and this is important.)
From State 9 there is a loop that goes through 32 states and then comes back to state 9. So far this NFA accepts multiples of 33. But we then put in a shortcut so that from state 31 go straight to state 9. We now see that a^n is accepted iff

$$n = 32x + 33y + 9$$

By the Corollary, this NFA has the behaviour we want.

End of Proof

3 The Smaller Loops

Theorem: There exist an NDFA M' on 25 states such that the following is true:

1. For all $n \le 999$, M accepts a^n.
2. M does not accepts a^{1000}.
3. We have NO COMMENT on the behavior of M on a^i for $i \ge 1001$.

Proof:

The NFA has a start state and then an e-transition to four different NFAs which we describe.

1. A 4-loop such that we accept all strings a^n such that $n \not\equiv 0 \pmod 4$, which we rewrite as $n \not\equiv 1000 \pmod 4$.
2. A 5-loop such that we accept all strings a^n such that $n \not\equiv 0 \pmod 5$, which we rewrite as $n \not\equiv 1000 \pmod 5$.
3. A 7-loop such that we accept all strings a^n such that $n \not\equiv 6 \pmod 7$, which we rewrite as $n \not\equiv 1000 \pmod 7$.
4. A 9-loop such that we accept all strings a^n such that $n \not\equiv 1 \pmod 9$, which we rewrite as $n \not\equiv 1000 \pmod 9$.

Let $n \le 1000$ and a^n rejected by M'. We show that $n = 1000$.
Since a^n is rejected all four branches reject it. Hence
$n \equiv 1000 \pmod 4$.
$n \equiv 1000 \pmod 5$.
$n \equiv 1000 \pmod 7$.
$n \equiv 1000 \pmod 9$.
We leave it to the reader to show that, given the above,
$n \equiv 1000 \pmod{4 \times 5 \times 7 \times 9}$
$n \equiv 1000 \pmod{1260}$
Since $n \le 1000$ and $n \equiv 1000 \pmod{1260}$, $n = 1000$.

End of Proof

4 A Small NFA for $\{a^y : y \neq 1000\}$

The following NFA accepts L: from the start state there are 5 e-transitions, one to the Big Loop, the rest to the four small loops from the last theorem.

If a^n is rejected then

- From the big loop we know that $n \leq 1000$.
- From the four small loops we know that $n = 1000$.

The NFA has $1 + 41 + 4 + 5 + 7 + 9 = 67$ states.

We can make it a bit smaller. The big loop first went through 9 states. You could instead have the start state be the 9th state of the loop rather than the first. So we can get htis down to 58.

Can we do better? Unknown to science!