Solution to Spring 2019 Midterm 2019

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Problem 1

For each of the following languages say if it's REGULAR or NOT REGULAR. If REGULAR then draw a DFA or give a regular expression for it.

(NOTE — I REALLY WANT the DFA or REGULAR expression for it!!!!!!!!!!!!!!) If NOT REGULAR then prove it is not regular.

Alphabet is $\{a, b\}$ and $\mathrm{N} = \{0, 1, 2, \ldots\}$.

Added For Spring 2020 Class If not regular than is it CFL?

Problem 1a

$$L_1 = \{a^n b^m \mid n, m \in \mathrm{N}\}$$

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REGULAR This is just $a^* b^*$.
DFA omitted but its easy.

Problem 1b

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REGEX for it: $\{a, b\}^* ba\{a, b\}^*$.
Could also do a DFA for $a^* b^*$ and swap final and nonfinal states.

Problem 1c

$$L_3 = \{a^n b^m \mid n \geq 2m\}$$

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Let $w = a^{2n}b^n$. By ZW Pumping Lemma we can make sure the xy is within the a's, so the y is within the a's.

$w = xyz$.

$x = a^{n_1}$, $y = a^{n_2}$, $z = a^{n_3}b^n$ where: $n_2 \neq 0$.

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$x = a^{n_1}$, $y = a^{n_2}$, $z = a^{n_3}b^{2n}$ where: $n_1 + n_2 + n_3 = 2n$

We know that $\forall i \geq 0$, $xy^iz \in L$.

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Take $i = 0$: to get $a^{n_1+n_3}b^n$.

Since $n_2 \neq 0$, $n_1 + n_3 < n_1 + n_2 + n_3 < 2n$, so NOT in L.

Problem 1c, Possible Alternative

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YES. The WZ PL. This is just ZW backwards. Rather than force $|xy|$ to be small, you can instead force $|yz|$ to be small. Then you could force $|yz|$ to be within the b's.

We omit details, work out a proof of WZ theorem yourself and a proof that L_3 is not regular using it.

Problem 1c, CFL part

Back to our problem.

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We showed L_3 not regular. Is L_3 context free?

Problem 1c, CFL part

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We showed L_3 not regular. Is L_3 context free?
YES:
$S \rightarrow XY$
$Y \rightarrow aaYb$
$Y \rightarrow e$
$X \rightarrow aX$
$X \rightarrow e$

Problem 1d

$$L_4 = \{a, b\}^* - L_3.$$

(In other words, all strings that are NOT in L_3.)
NOT REGULAR If L_4 was regular than, since regular is closed under Comp, L_3 would be regular.

Problem 1d, CFL Part

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Is L_4 Context Free?

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Is L_4 Context Free?
YES. L_4 is the union of the following CFL's

1. $\{a, b\}^* ba \{a, b\}^a$ (This one is easy and we leave to you.)
2. $\{a^n b^m \mid n < 2m\}$. We discuss how this is CFL.

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$\{a^n b^m \mid n < 2m\}$ is the union of the following CFL's

1. $\{a^n b^m \mid n \le m\}$.
2. $\{a^n b^m \mid m \le n \le 2m\}$.

Problem 1d, CFL Part, Cont

CFL for $\{a^n b^m \mid n \leq m\}$.

$S \rightarrow XY$
$X \rightarrow aSb$
$Y \rightarrow bY$
$Y \rightarrow e$

Problem 1d, CFL Part, Cont

CFL for $\{a^n b^m \mid m \leq n \leq 2m\}$.

$S \rightarrow aSb$
$S \rightarrow aSbb$
$S \rightarrow e$

Problem 4

Let $\Sigma = \{a, b\}$.
L_1 is regular via DFA $M_1 = (Q_1, \Sigma, \delta_1, s_1, F_1)$,
L_2 is regular via DFA $M_2 = (Q_2, \Sigma, \delta_2, s_2, F_2)$.
Note that \$ is NOT in Σ. Construct a DFA for:

$$\{x\$y \mid x \in L_1 \text{ and } y \notin L_2\}$$

using M_1 and M_2. Briefly describe how your DFA is constructed. (NOTE- we are asking for a construction, not a drawing. Recall we did constructions with DFA's to prove that if L_1 and L_2 are regular than so are BLAH.)

NOTE- Do NOT give an NFA and say *we can convert it to a DFA*. I want the DFA!
REMINDER: Your DFA must have, for every state q and alphabet symbol σ, $\delta(q, \sigma)$ defined.

Problem 4 SOLUTION

$(Q_1 \cup Q_2 \cup \{\text{DUMP}\}, \Sigma \cup \{\$\}, \delta, s_1, Q_2 - F_2)$
Where δ is defined as follows:
If $q \in Q_1$ and $\sigma \in \{a, b\}$ then $\delta(q, \sigma) = \delta_1(q, \sigma)$.
If $q \in Q_1 - F_1$ then $\delta(q, \$) = \text{DUMP}$
If $q \in F_1$ then $\delta(q, \$) = s_2$
If $q \in Q_2$ and $\sigma \in \{a, b\}$ then $\delta(q, \sigma) = \delta_2(q, \sigma)$.
If $q \in Q_2$ then $\delta(q, \$) = \text{DUMP}$.
$\delta(\text{DUMP}, \sigma) = \text{DUMP}$ for $\sigma \in \{a, b, \$\}$.

What Some Students Tried To Do

- Some students tried to do a cross product construction. I suspect they did the one thing I told students $\aleph_0$ times: *do not memorize things* and *not copy things off of your cheat sheet*. These students got 0.
- There was a mild ambiguity in the problem. We had $y \notin L_2$. Do we mean $\{a, b\}^* - L_2$ or $\{a, b, \$\}^* - L_2$. We were happy with either interpretation. Almost everyone interpreted it as $\{a, b\}^* - L_2$.
- Some students did not have a dead state. We let this one pass. We will not be so generous in the future. Why did you leave it out when I said explicitly that every $\delta(q, \sigma)$ has to be defined? These are the questions that try ones soul.
- Some students copied the $L_1 L_2$ construction, so they had e instead of \$ for going from M_1 to M_2. Note that this is NOT a DFA. And they are likely copying from their cheat sheets. They got ZERO.

Problem 5

For this problem you may use the following theorem.
Theorem: If x, y are relatively prime then

- For all $z \geq xy - x - y + 1$ there exists $c, d \in \mathbb{N}$ such that $z = cx + dy$.
- There is no $c, d \in \mathbb{N}$ such that $xy - x - y = cx + dy$.

The alphabet is $\{a\}$. Let

$$L = \{a^n \mid n \neq 117\}$$

1. (10 points) Does there exist a DFA for L with < 120 states? If so then draw the DFA; you may use DOT DOT DOT. If not then PROVE there is no such DFA.
2. (10 points) Does there exist an NFA for L with less than 60 states? If so then draw the NFA; you may use DOT DOT DOT. If not then PROVE there is no such NFA.

Problem 5 Solution

We just Sketch this one since we've done so many like it.
Thought Process: First take $x = \lceil \sqrt{117} \rceil = 11$ and $y = 12$. Then

$$xy - x - y = 89$$

Rather small. Lets see if $x = 12$, $y = 13$ still gives us ≤ 117

$$xy - x - y = 109$$

1. $x = 12$, $y = 13$, Tail 117-109=8. $8 + 13 = 21$ states.
2. Primes: $2 \times 3 \times 5 \times 7 = 210 > 117$, $2 + 3 + 5 + 7 = 17$ states.

Total number of states: 21+17=38.
Note On An Exam you would have to do the whole problem, give the NFA or a picture of it.
Note Can do better?

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Note Can do better? YES: use $4 \times 5 \times 7 = 140 > 117$, $4 + 5 + 7 = 16$, so $21 + 16 = 37$.

Extra Problems to Think About

Definition If $w = \sigma_1 \cdots \sigma_n$ is a string then any string of the form

$$\sigma_{i_1} \cdots \sigma_{i_k}$$

where $i_1 < \cdots < i_k$ is a *subsequence of* w.
$SUBSEQ(w)$ is the set of all subsequences of the string w.
Examples If $w = aaba$ then the subsequences are
$SUBSEQ(aaba) = \{e, a, b, aa, ab, ba, aaa, aab, aba, aaba\}$.
Definition If $L \subseteq \{a, b\}^*$ then

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T or F and prove:

1. If L is regular than $SUBSEQ(L)$ is regular.
2. If L is context free than $SUBSEQ(L)$ is context free.