Decidability and Undecidability

Exposition by William Gasarch—U of MD

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4. Everything computable is computable by some TM.
5. A TM that halts on all inputs is called **total** .

Computable Sets

Def A set A is *computable* if there exists a Turing Machine M that behaves as follows:

$$M(x) = \begin{cases} Y & \text{if } x \in A \\ N & \text{if } x \notin A \end{cases} \tag{1}$$

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Notation DEC is the set of Decidable Sets.

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2. Yes—ALL SETS: uncountable. DEC Sets: countable, hence there exists an uncountable number of noncomputable sets.
3. That last answer is true but unsatisfying. We want an actual example of an noncomputable set.

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Recall You all thought there was no small NFA for $\{a^i : i \neq n\}$ and were wrong. Hence lower bounds need proof.

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$M_e(e)\downarrow \implies M(e,e) = Y \implies M_e(e)\uparrow$
$M_e(e)\uparrow \implies M(e,e) = N \implies M_e(e)\downarrow$
We now have that $M_e(e)$ cannot $\downarrow$ and cannot $\uparrow$. **Contradiction.**

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Proofs by reductions. Similar to NPC. We **will not** do that.

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   Is this interesting? **No** Machines related to other machines.

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   A pedagogical nightmare!

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Key Difference:

- **Semantic Question** : What does M_e do? is usually undecidable.
- **Syntactic Question** : What does M_e look like? is usually decidable.

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Does this definition remind you of something? YES- NP.

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   My thesis was on showing some of those limits.

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Because of (3) Σ_1 is often called **recursively enumerable** or **computably enumerable** .

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TOT is **harder** than HALT.

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Can we get this lower in the Arithmetic Hiearchy? Vote Yes. The first quanfier is over a finite set. So better:

Lower in the Arith Hierarchy

$$\{e : (\exists x_1, \ldots, x_{e-1}, s)(\forall t)$$

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Yes—a few. we will discuss three.

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Hilbert thought this would inspire interesting Number Theory.

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But they didn't have Hilbert's Tenth Problem undecidable... yet.

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The proof involved coding Turing Machines into Polynomials.

Upshot This problem of, given $p(x_1, \ldots, x_n) \in \mathbb{Z}[x_1, \ldots, x_n]$ does it have an integer solution is a natural question that is undecidable.

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Math (and the rest of life) is full of stories of jealousy and credit-claimers (e.g., Newton vs Leibnitz) so its interesting that this aspect is boring.

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Back to Math

Hilbert's 10th problem (in modern language) Give an algorithm that will, given $p(x_1, \ldots, x_n) \in \mathbb{Z}[x_1, \ldots, x_n]$ determine if there exists $a_1, \ldots, a_n \in \mathbb{Z}$ such that $p(a_1, \ldots, a_n) = 0$.

We now know this is undeciable.
For which degrees d and number-of-vars n is it undeciable? Decidable?
For a full account see Gasarch's survey `h10.pdf`
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Proof involves looking at the set of all accepting sequences of configurations.
(We will not be doing that, but the proof is here:
`https://www.cs.umd.edu/users/gasarch/COURSES/452/S20/notes/undcfg.pdf`

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