BILL, RECORD LECTURE!!!!

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The Complexity of Problems: P and NP

Exposition by William Gasarch—U of MD

Complexity of Sets

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To even ask these questions we need (1) a standard way to describe sets and a (2) model of computation.

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3. To define **Algorithm** we need a model of computation.

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2. Turing machines compute with discrete steps so one can talk about how many steps a computation takes.
3. There are many models of computation. They are all equiv up to **poly time.** Hence **poly time** can be defined without getting into the details of a Turing machine or other models.

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1. $\mathrm{SAT} \in \mathrm{EXP}$, time 2^n, by brute force.
2. If I came up with a $(1.5)^n$ algorithm that's **just brute force** with some tricks.
3. If I came up with an n^{1000} algorithm then it's **NOT brute force**. I would have found something **very clever**. Not practical, but that cleverness can probably be exploited to get a practical algorithm.

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3. Quadratic time not closed under composition: if $f(n), g(n)$ are quadratic then $f(g(n))$ is quartic, not quadratic.
4. P is closed under composition: if $f(n), g(n)$ are poly then $f(g(n))$ is poly.

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SAT, HAM, EUL are similar.

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- If $x \notin A$ then there is NO proof that $x \in A$.

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    - None of SAT, HAM, CLIQ are in P.

Reductions

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Contrapositive If $X \leq Y$ and $X \notin \mathrm{P}$ then $Y \notin \mathrm{P}$.

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The condition:

$$\text{for EVERY } X \in \mathrm{NP},\ X \leq Y?$$

seemed very hard to meet.

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3. $\mathrm{3SAT}$ is CNF-SAT where each clause has ≤ 3 literals.

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3. Thousands of problems are NP-complete. If any are in P then they are all in P.
4. Most Computer Scientists and Mathematicians think $\mathrm{P} \neq \mathrm{NP}$.

History: HAM and EUL

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They didn't have the notion of algorithms to state what they wanted more rigorously.

The theory of NP-completeness enabled mathematicians to **state** what they wanted rigorously ($\mathrm{HAM} \in \mathrm{P}$) and also gave the basis for proving likely it **cannot** be done (since HAM is NP-Complete).

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2. CLIQ is NP-complete. We prove that on next few slides.
3. HAM is NP-complete. Just take my word for it.

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2) Graph G with $7k$ vertices as follows: For each clause we have 7 vertices. Label them with the 7 ways to set the 3 vars to make the clause satisfiable. For example, for the clause $x \vee y \vee \neg z$, we have 7 vertices: TTT, TTF, TFT, TFF, FTT, FTF, FFF.

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There are no edges between vertices associated to the same clause. We put an edge between vertices associated with different clauses if the assignments do not conflict. Example:
$(x = T, y = T, z = T)$ has edge to $(w = F, x = T, z = T)$ but not to $(w = F, x = F, z = T)$.

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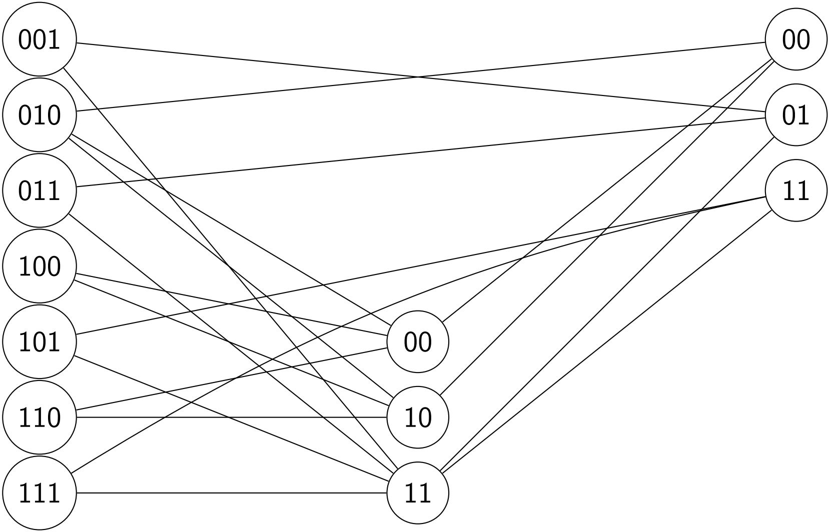
3) Example on next slide

$$(x \vee y \vee z) \quad \wedge \quad (w \vee \overline{z}) \quad \wedge \quad (\overline{x} \vee z)$$



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2. We **do not know** that $\mathrm{CLIQ} \notin \mathrm{P}$.
3. We **do know** that $\mathrm{3SAT} \in \mathrm{P}$ IFF $\mathrm{CLIQ} \in \mathrm{P}$.
4. We **believe** $\mathrm{3SAT} \notin \mathrm{P}$, hence we **believe** $\mathrm{CLIQ} \notin \mathrm{P}$.

Why Do We Believe $\mathrm{P} \neq \mathrm{NP}$?

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3. Intuitively **coming up with a proof** seems harder than **verifying a proof**.
4. $\mathrm{P} \neq \mathrm{NP}$ has great explanatory power. See next slide.

Approximating Set Cover

Set Cover Given n and $S_1, \ldots, S_m \subseteq \{1, \ldots, n\}$ find the least number of sets S_i's that **cover** $\{1, \ldots, n\}$.

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3. These two proofs have nothing to do with each other yet give matching upper and lower bounds.
4. There are many other approx problems where $\mathrm{P} = \mathrm{NP}$ explains why they cannot be improved.

End with some Opinions

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   1.2 IF $\mathrm{P} \neq \mathrm{NP}$ this will not be proven until the year 2525.
2. $\mathrm{P} \neq \mathrm{NP}$. In fact, SAT requires $2^{\Omega(n)}$ time.

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