Homework 11 Morally Due May 7 at 3:30PM

1. (25 points) In this problem we will guide you through a proof that the following problem is undecidable:

   Given a CFG G, is $\overline{L(G)}$ a CFL? We call this problem CFG-COMP.

   We define a variant of $\mathrm{ACC}_{e,x}$.

   Def For $e \in \mathsf{N}$, ACC_e is the set of all sequences of config's represented by

   $$\$C_1\$C_2^R\$C_3\$C_4^R\$\cdots\$C_s^R\$$$

   such that

   - $|C_1| = |C_2| = \cdots = |C_s|$.
   - There is an x such that $C_1, C_2, \ldots, C_s$ represents an accepting computation of $M_e(x)$.

   (a) (5 points) Show that $\overline{ACC_e}$ is a context free grammar. (You only have to modify ONE of the parts of the CFG description for $\overline{\mathrm{ACC}_{e,x}}$. The one having to do with the initial configuration. You do NOT have to rewrite the entire rest of the proof.)

   (b) (0 points, so don't hand it in, just think about it.) Show that if M_e accepts an INFINITE number of x then ACC_e is NOT a CFL. (You may use the PL for CFG's.)

   (c) (10 points) Show that if M_e accepts a FINITE number of x then ACC_e IS a CFL.

   (d) (10 points) Let INF be the set of Turing machines that accept an INFINITE number of inputs. It is known that INF is undecidable (indeed!- its in $\Pi_2 - \Sigma_1$). You may use this fact.

   Show that if CFG-COMP is decidable then INF is decidable. We give you some of the algorithm and BLANKS. You need to fill in the BLANKS.

   - Input e. Create a CFG G for $\overline{\mathrm{ACC}_e}$.
   - BLANK

   If $e \in$ INF then BLANK hence the algorithm says YES.

   If $e \notin$ INF then BLANK hence the algorithm says NO.

2. (30 points- 10 points each)

   (a) Let $a_1, \ldots, a_k$ and $m_1, \ldots, m_k$ be natural numbers such that $(\forall i)[0 \le a_i \le m_i]$.
   Show that the following set is Diophantine. What is the degree of the polynomial? What is the number of variables (including x).

$$\left\{ x : \bigwedge_{i=1}^{k} x \equiv a_i \pmod{m_i} \right\}.$$

   (b) Let $a_1, \ldots, a_k$ and $m_1, \ldots, m_k$ be natural numbers such that $(\forall i)[0 \le a_i \le m_i]$.
   Show that the following set is Diophantine. What is the degree of the polynomial? What is the number of variables (including x).

$$\left\{ x : \bigvee_{i=1}^{k} x \equiv a_i \pmod{m_i} \right\}.$$

   (c) Let $p_1, \ldots, p_k$ be the first k primes. Show that the following set is Diophantine. What is the degree of the polynomial? What is the number of variables (including x).

$$\left\{ x : \bigwedge_{i=1}^{k} x \not\equiv 0 \pmod{p_i} \right\}.$$

3. (20 points) Read or re-read the last part of the $(Q, <)$ slides– the part about the horse number. The numbers $H(n)$ will come up in this problem.

   For $n \geq 2$. Let $B(n)$ be the number of ways that n horses, $x_1, \ldots, x_n$, can finish a race (equalities allowed) such that $x_1 < x_2$.

   (a) (0 points but you should do it or convince yourself that you could). What is $B(2)$, $B(3)$, $B(4)$. Do $B(4)$ in such a way that it can be generalized to $B(n)$.

   (b) (20 points) Give a recurrence for $B(n)$. It may also involve $H(n)$. For example, it could be (but its NOT) $B(n) = B(n-1) + B(n-4) + H(n)H(n-3)$. Explain your answer.

4. (25 points) In this problem if G is a CFL then $L(G)$ is the set of strings that G generates.

   In this problem the alphabet is $\{a, b\}^*$.

   (a) (10 points) Show that there is a CFL G in Chomsky normal form with $L(G) = \{a^n\}$ such that G has $O(\log n)$ rules. (You may assume n is a power of 2.)

   (b) (15 points) Let w be a Kolmogorov random string of length n. Think of n as large.

   Let G be a CFL in Chomsky Normal Form such that $L(G) = \{w\}$. Show that Then G has at least $\Omega(n^{0.9})$ rules. (Note: A better result is possible; however, we are asking for $\Omega(n^{0.9})$ to keep the algebra simpler.)

   Hint If a CFL has R rules then it has at most $3R$ nonterminals. In this case each nonterminal can be represented with $O(\log R)$ bits. Hence the size of the CFL is $O(R \log R)$ bits.