CLIQ $\le$ SAT

Exposition by William Gasarch—U of MD

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Bill Because there are **awesome SAT Solvers!**

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1. SAT solvers are only good on some problems.
2. Getting the reductions to not blow up is not always possible.

How to View CLIQ

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G has a clique of size k is **equivalent** to:
There is a 1-1 function $\{1, \ldots, k\} \rightarrow V$ such that for all $1 \le a, b \le k$, $(f(a), f(b)) \in E$.

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Intent

$$x_{ij} = \begin{cases} T & \text{if numb } i \text{ maps to vertex } j \\ F & \text{if numb } i \text{ does not maps to vertex } j \end{cases} \qquad (1)$$

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Note So far all we've used about G is that it has n vertices.

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- Upshot: probably really good on sparse graphs.