Decidability and Undecidability

Exposition by William Gasarch—U of MD

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- 4. Everything computable is computable by some TM.
- 5. A TM that halts on all inputs is called total.

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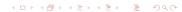
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Some examples of computable sets.

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- 4. $\{e: M_e \text{ has a prime number of states }\}$.

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- Yes—ALL SETS: uncountable. DEC Sets: countable, hence there exists an uncountable number of noncomputable sets.
- 3. That last answer is true but unsatisfying. We want an actual example of an noncomputable set.

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Recall You all thought there was no small NFA for $\{a^i : i \neq n\}$ and were wrong. Hence lower bounds need proof.



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We now have that $M_e(e)$ cannot \downarrow and cannot \uparrow . Contradiction.

Using that HALT is undecidable we can prove the following undecidable:

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 A pedagogical nightmare!

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Key Difference:

- ► Semantic Question : What does M_e do? is usually undecidable.
- ► Syntactic Question : What does M_e look like? is usually decidable.

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- 2.1 For NP easy means P and the quant is over an exp size set.
 2.2 For Σ₁ easy means DEC and the quant is over N.
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- 4. Are ideas from Computability theory useful in complexity theory?

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My thesis was on showing some of those limits.

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Because of (3) Σ_1 is often called **recursively enumerable** or **computably enumerable** .

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Def *B* is always a decidable set. $A \in \Pi_1$ if $A = \{x : (\forall y)[(x,y) \in B]\}$. $A \in \Sigma_2$ if $A = \{x : (\exists y_1)(\forall y_2)[(x,y_1,y_2) \in B]\}$. $A \in \Pi_2$ if $A = \{x : (\forall y_1)(\exists y_2)[(x,y_1,y_2) \in B]\}$.

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TOT is harder than HALT.
```

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Set of Turing Machines that compute increasing functions:

$$\{e: (\forall x < y)(\exists s)[M_{e,s}(x) \downarrow < M_{e,s}(y) \downarrow]\} \in \Pi_2.$$

Set of Turing machines that halt on all but a finite number of inputs

$$\{e: (\exists x)(\forall y > x)(\exists s)[M_{e,s}(y)\downarrow].$$

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It may have taken people outside of the mathemmatical mainstream to even think the problem was undecidable. But they didn't have Hilbert's Tenth Problem undecidable... yet.

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The proof involved coding Turing Machines into Polynomials.

Upshot This problem of, given $p(x_1, ..., x_n) \in \mathbb{Z}[x_1, ..., x_n]$ does it have an integer solution is a natural question that is undecidable.

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Math (and the rest of life) is full of stories of jealousy and credit-claimers (e.g., Newton vs Leibnitz) so its interesting that this aspect is boring.

Hilbert's 10th problem (in modern language) Give an algorithm that will, given $p(x_1, \ldots, x_n) \in \mathbb{Z}[x_1, \ldots, x_n]$ determine if there exists $a_1, \ldots, a_n \in \mathbb{Z}$ such that $p(a_1, \ldots, a_n) = 0$.

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- 5. Dec with deg- ∞ , vars-1. Easy.

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- 7. Dec with deg-2, vars- ∞ . Hard. Recent (1972).



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Answer on next slide.

1. Easy to show that if $k \equiv 4,5 \pmod{9}$ then NO solution. All future items assume that restriction.

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- 2. If $k \le 1000$, k not on list below, and $\max\{|x|,|y|,|z|\} \le 10^{15}$ then k is sum of three cubes.

114, 164, 390, 579, 627, 633, 732, 921, 975

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- 5. LARGE knowledge gap between decidable and undecidable.

Input $n \in \mathbb{N}$ and a set $\{M_1, \dots, M_m\}$ of $n \times n$ matrices over \mathbb{Z} .

Input $n \in \mathbb{N}$ and a set $\{M_1, \ldots, M_m\}$ of $n \times n$ matrices over \mathbb{Z} . Question Does some product of the matrices equal the ZERO matrix? (You can use a matrix more than once.)

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Everything elseis unknown to science . We pick out two:

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Everything elseis unknown to science . We pick out two:

1. For two 3×3 matrices, unknown.

Input $n \in \mathbb{N}$ and a set $\{M_1, \ldots, M_m\}$ of $n \times n$ matrices over \mathbb{Z} . Question Does some product of the matrices equal the ZERO matrix? (You can use a matrix more than once.)

This problem is undecidable. We refine this:

- 1. For two 15×15 matrices, undecidable.
- 2. For three 9×9 matrices, undecidable.
- 3. For four 5×5 matrices, undecidable.
- 4. For six 3×3 matrices, undecidable.
- 5. For two 2×2 matrices, decidable.

Everything elseis unknown to science . We pick out two:

- 1. For two 3×3 matrices, unknown.
- 2. For three 2×2 matrices, unknown.

No

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Proof involves looking at the set of all accepting sequences of configurations.

(We will not be doing that, but the proof is here: https://www.cs.umd.edu/users/gasarch/COURSES/452/S20/notes/undcfg.pdf

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- (1) Given $p \in \mathbb{Z}[x_1, \dots, x_n]$ does p have an integer solution?
- (2) Given Matrices M_1, \ldots, M_m , does some product = ZERO?
- (3) Given a CFG G, is $\overline{L(G)}$ a CFL?