**Homework 6 Morally Due March 12 at 3:30PM**

1. (0 points, but if you actually miss the midterm without telling Dr. Gasarch ahead of time, you will lose 100 points on this homework) When will the midterm be (give date and time)?

2. (25 points) In class I showed a protocol for EQ that used $O(\log n)$ bits but had an error of $\sim \frac{1}{n}$.

   (a) (10 points) Sam DEMANDS that we get a protocol that has error of $\sim \frac{1}{n^2}$ and still uses $\ll n$ bits. Give such a protocol. Prove that the error rate is $\sim \frac{1}{n^2}$. State how many bit are used. Use big O notation.

   (b) (15 points) Let $k(n)$ be a very slow growing function of $n$ (so perhaps $k = \log\log n$). Now Sam DEMANDS that we get a protocol that has error of $\sim \frac{1}{n^{k(n)}}$ and still uses $\ll n$ bits. Give such a protocol. Prove that the error rate is $\sim \frac{1}{n^{k(n)}}$. State how many bit are used. Use big O notation.

   **SOLUTION**

   We do a protocol with error $\sim \frac{1}{n^{L-1}}$.

   For the first problem plug in $L = 3$.

   For the second problem plug in $L = k(n) + 1$.

   i. Alice has $a_0 \cdots a_{n-1}$, Bob has $b_0 \cdots b_{n-1}$.
   ii. Alice sends a prime $p$ such that $n^L \le p \le 2n^L$.
   iii. Alice picks $z \in \{0, \ldots, p\}$ randomly.
   Alice computes, mod $p$,
   $y = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1}$.
   Alice sends $(z, y)$ to Bob.
   iv. Bob computes, mod $p$,
   $y' = b_0 + b_1 z + \cdots + b_{n-1} z^{n-1}$.
   If $y = y'$ then send 1, else send 0.

   The prob of error is the prob that $z$ is a rood of $a(x) - b(x) = 0$.

   There are $n$ roots. Their domain $\{0, \ldots, p-1\}$ is of size $p \sim n^L$.

   Hence the error is $\sim \frac{n}{n^L} = \frac{1}{n^{L-1}}$.

   If $L = 3$ you get error $\frac{1}{n^2}$.

   If $L = k(n) + 1$ you get error $\frac{1}{n^{k(n)}}$.

   The number of bits is the number of bits for the numbers $p$ and $y$.

   Since $n^L \le p, y \le 2n^L$, both take $\le \log(2n^L) = O(L \log n)$ bits.

   If $L = 3$ you get the number of bits is error $O(\log n)$.

   If $L = k(n) + 1$ you get the number of bits is $O(k(n) \log n)$.

   **END OF SOLUTION**

3. (25 points)

   In this problem we guide you the the proof that $f(45,26) \le \frac{32}{78}$.

   Assume, by way of contradiction, that there is a $(45,26)$-procedure with smallest piece $> \frac{32}{78}$.

   By the usual techniques, we can assume that every muffins is cut into exactly 2 pieces. Hence there are 90 pieces.

   **Case 1:** Alice gets $\ge 5$ shares. SHOW HOW THIS LEADS TO SOME SHARE BEING of size $< \frac{32}{78}$.

   **Case 2:** Bob gets $\le 2$ shares. SHOW HOW THIS LEADS TO SOME SHARE BEING of size $< \frac{32}{78}$.

   In the subsequent cases we assume the negation of Cases 1 and 2. Hence every student either gets 3 shares or 4 shares.

   A student who gets 3 shares is called a 3-student.

   A student who gets 4 shares is called a 4-student.

   A share that is given to a 3-student is called a 3-share.

   A share that is given to a 4-student is called a 4-share.

   Let $s_3$ (resp. $s_4$) be the number of 3-students (resp. 4-students). SHOW HOW YOU DEDUCE there are 14 3-students and 12 4-students. Note that there are 42 3-shares, and 48 4-shares.

   We now look at intervals.

   **Case 3:** Alice has a 4-share $\ge \frac{39}{78}$. SHOW HOW THIS LEADS TO SOME SHARE BEING of size $\le \frac{32}{78}$.

   **Case 4:** Bob has a 3-share $\le \frac{43}{78}$. SHOW HOW THIS LEADS TO SOME SHARE BEING of size $\le \frac{32}{78}$.

   **Case 5:** The following picture captures the negation of cases 1,2,3, and 4.

| | | | | | | |
|---|---|---|---|---|---|---|
| ( | 48 4-shs | )[ | 0 | ]( | 42 3-shs | ) |
| $\frac{32}{78}$ | | $\frac{39}{78}$ | | $\frac{43}{78}$ | | $\frac{46}{78}$ |

   SHOW HOW WE GET A CONTRADICTION.

*Hint:* Note that $\frac{39}{78} = \frac{1}{2}$.

END OF PROOF

**SOLUTION**

In this problem we guide you the the proof that $f(45,26) \le \frac{32}{78}$.

Assume, by way of contradiction, that there is a $(45,26)$-procedure with smallest piece $> \frac{32}{78}$.

By the usual techniques, we can assume that every muffins is cut into exactly 2 pieces.

**Case 1:** Alice gets $\ge 5$ shares. Then one of them is $\le \frac{45}{26} \times \frac{1}{5} = \frac{27}{78} < \frac{32}{78}$.

**Case 2:** Bob gets $\le 2$ shares. Then one of the shares is $\ge \frac{45}{26} \times \frac{1}{2} = \frac{67.5}{78}$. Its buddy is $\le 1 - \frac{67.5}{78} = \frac{10.5}{78} < \frac{32}{78}$.

In the subsequent cases we assume the negation of Cases 1 and 2. Hence everyone is either a 3-student or a 4-student. Let $s_3$ (resp. $s_4$) be the number of 3-students (resp. 4-students). Since there are 90 pieces and 26 students,

$$\begin{aligned} 3s_3 + 4s_4 = &\ 90 \\ s_3 + s_4 = &\ 26. \end{aligned}$$

Hence $s_3 = 14$ and $s_4 = 12$. So there are fourteen 3-students, twelve 4-students, forty-two 3-shares, and forty-eight 4-shares. Since $48 > 45$, if all of the 4-shares are $< \frac{1}{2}$, that will be a contradiction (since at least half of the pieces must be $\ge \frac{1}{2}$). Indeed, this will be our contradiction.

We now look at intervals.

**Case 3:** Alice has a 4-share $\ge \frac{39}{78}$. Alice's other three 4-shares add up to $\le \frac{135}{78} - \frac{39}{78} = \frac{96}{78}$, hence one of them is $\le \frac{96}{78} \times \frac{1}{3} = \frac{32}{78}$.

**Case 4:** Bob has a 3-share $\le \frac{43}{78}$. Bob's other two 3-shares add up to $\ge \frac{135}{78} - \frac{43}{78} = \frac{92}{78}$, hence one of the shares is $\ge \frac{92}{78} \times \frac{1}{2} = \frac{46}{78}$. Its buddy is $\le 1 - \frac{46}{78} = \frac{32}{78}$.

**Case 5:** The following picture captures the negation of cases 1,2,3, and 4.

| ( | 48 4-shs | )[ | 0 | ]( | 42 3-shs | ) |
|---|---|---|---|---|---|---|
| $\frac{32}{78}$ | | $\frac{39}{78}$ | | $\frac{43}{78}$ | | $\frac{46}{78}$ |

The midpoint is $\frac{1}{2} = \frac{39}{78}$. Note that all forty-eight 4-shares are $< \frac{1}{2}$. This is a contradiction.

END OF PROOF

**END OF SOLUTION**

4. (25 points)

   (a) (10 points) Proof without using the Floor-Ceiling theorem that any protocol that takes 10 muffins and divides them for 3 people so that everyone gets $\frac{10}{3}$ has to have a piece of size $\le \frac{4}{9}$. (So $f(10,3) \le \frac{4}{9}$.) You can assume that in any such protocol every muffins is cut into exactly 2 pieces.

   (b) (15 points) Give a protocol that takes 10 muffins and divides them for 3 people so that everyone gets $\frac{10}{3}$, and every piece is of size $\ge \frac{4}{9}$ . (So $f(10,3) \ge \frac{4}{9}$.)

**SOLUTION**

PART (a): $f(10,3) \le \frac{4}{9}$.

Assume there is a protocol for $(10,3)$. Since every muffin is cut into 2 pieces there are 20 pieces. Since there are 3 people BOTH of the following happen

- Some student gets $\ge \lceil \frac{20}{3} \rceil = 7$ pieces. Then one of her pieces is $\le \frac{10}{3} \times \frac{1}{7} = \frac{10}{21}$. SO some piece is $\le \frac{10}{21}$. NOT helpful!
- Some student gets $\le \lfloor \frac{20}{3} \rfloor = 6$ pieces. Then one of her pieces is $\ge \frac{10}{3} \times \frac{1}{6} = \frac{5}{9}$. Look at the muffin that piece came from. The other part of that muffin is of size $\le 1 - \frac{5}{9} = \frac{4}{9}$. GREAT!

PART (b): $f(10,3) \ge \frac{4}{9}$.

(a) Cut 6 muffins $(\frac{4}{9}, \frac{5}{9})$.

(b) Cut 4 muffins $(\frac{1}{2}, \frac{1}{2})$.

(c) Give one student 6 pieces of size $\frac{5}{9}$. That's $\frac{30}{9} = \frac{10}{3}$.

(d) Give two students 3 pieces of size $\frac{4}{9}$ and 4 pieces of size $\frac{1}{2}$. That's $3 \times \frac{4}{9} + 4 \times \frac{1}{2} = \frac{4}{3} + 2 = \frac{10}{3}$.

**END OF SOLUTION**

5. (25 points)

(a) (15 points) Consider the formula

$(x_1 \vee \neg x_2)\wedge$
$(x_2 \vee \neg x_3)\wedge$
$(x_3 \vee \neg x_4)\wedge$
$\vdots$
$(x_{n-1} \vee \neg x_n)\wedge$
$(x_n \vee \neg x_1)$

How many satisfying assignments does this formula have? Justify! (Note that it may be a function of $n$.)

**SOL**

Recall that $p \rightarrow q$ is equivalent to $p\vee \not q$. Hence we can rewrite the formula as

$(x_1 \implies x_2)\wedge$
$(x_2 \implies x_3)\wedge$
$(x_3 \implies x_4)\wedge$
$\vdots$
$(x_{n-1} \implies x_n)\wedge$
$(x_n \implies x_1)$

If $x_1 = T$ then this forces $x_2 = T$, ..., $x_n = T$. So one satisfying assignment is $(T, \ldots, T)$.

If $x_1 = F$ then this forces $x_n = F$, ..., $x_2 = F$. So one satisfying assignment is $(F, \ldots, F)$.

Since either $x_1 = T$ or $x_1 = F$, there are exactly two satisfying assignments.

**END OF SOL**

(b) (10 points) Let $n \geq 4$. Give an example of a 2CNF formula which uses $n$ variables and is NOT satisfiable. (You may assume $n$ is even or odd or anything of that sort to make the answer smoother.)

**SOL**

We use $x_1, \ldots, x_n$.

We will assume $n$ is even.

$$(x_1 \vee x_2) \wedge (\neg x_1 \vee x_2) \wedge (x_1 \vee \neg x_2) \wedge (\neg x_1 \vee \neg x_2) \wedge$$

$$(x_3 \vee x_4) \wedge (x_5 \vee x_6) \wedge \cdots \wedge (x_{n-1} \vee x_n)$$

note that since the last var of each clause has an even index, it was good that $n$ is even.

**END OF SOL TO 1d**