HW10 Solution

Image of a $<$-Increasing Comp Funct is Dec

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   - Find x with $f(x) < y < f(x+1)$. Then output NO and halt.

Is the image of a $\leq$-Increasing Comp Funct Dec?

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Answer on next page.

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Bill I am absolutely okay with that.

H10 for Polys in One Var

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Formal Alg on Next Slide.

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2. If $a_0 = 0$ then output YES and halt (since 0 will be a root).
3. (If here then $a_0 \neq 0$) Let $d_1, \ldots, d_L$ be ALL divisors of $-a_0$. (**KEY** there are a finite number of them.)

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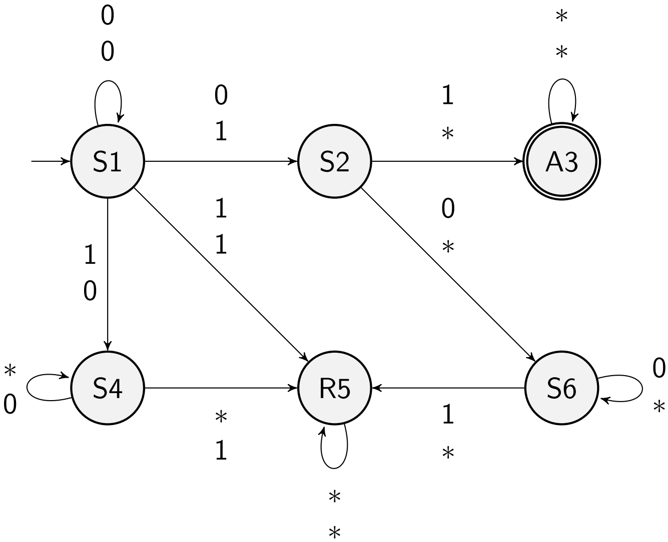
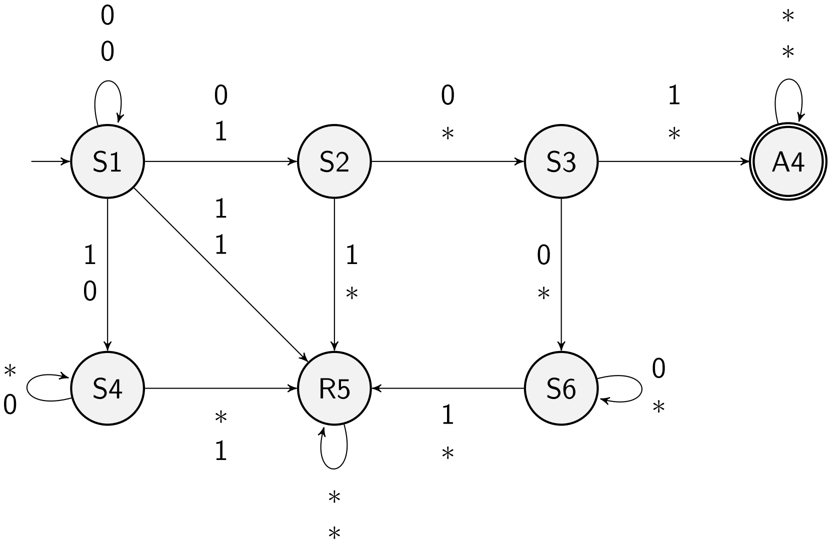
5. (If you got here then none of $p(d_i) = 0$.) Output NO.

DFA for WS1S Formula $x = y + 1$



DFA for WS1S Formula $x = y + 2$



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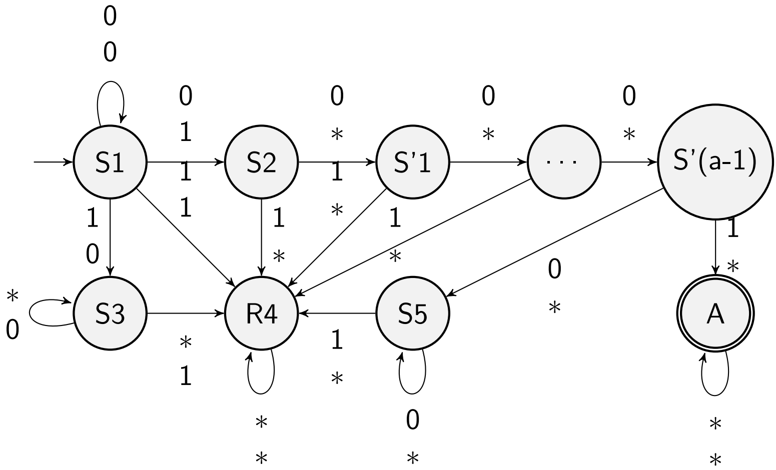

Takes $a + 5$ states