Primitive Recursive Functions

Exposition by William Gasarch-U of MD

Definition of Primitive Recursive (PR)

Def $f(x_1, \ldots, x_n)$ is **PR** if either:

Definition of Primitive Recursive (PR)

Def $f(x_1, \dots, x_n)$ is **PR** if either:

1. $f(x_1, \dots, x_n) = 0$;

Definition of Primitive Recursive (PR)

Def $f(x_1, \ldots, x_n)$ is **PR** if either:

1. $f(x_1, \ldots, x_n) = 0$;
2. $f(x_1, \ldots, x_n) = x_i$;

Definition of Primitive Recursive (PR)

Def $f(x_1, \ldots, x_n)$ is **PR** if either:

1. $f(x_1, \ldots, x_n) = 0$;
2. $f(x_1, \ldots, x_n) = x_i$;
3. $f(x_1, \ldots, x_n) = x_i + 1$;

Definition of Primitive Recursive (PR)

Def $f(x_1, \ldots, x_n)$ is **PR** if either:

1. $f(x_1, \ldots, x_n) = 0$;
2. $f(x_1, \ldots, x_n) = x_i$;
3. $f(x_1, \ldots, x_n) = x_i + 1$;
4. $g_1(x_1, \ldots, x_k)$, …, $g_n(x_1, \ldots, x_k)$,$h(x_1, \ldots, x_n)$ PR $\implies$

$$f(x_1, \ldots, x_k) = h(g_1(x_1, \ldots, x_k), \ldots, g_n(x_1, \ldots, x_k)) \text{ is PR}$$

Definition of Primitive Recursive (PR)

Def $f(x_1, \ldots, x_n)$ is **PR** if either:

1. $f(x_1, \ldots, x_n) = 0$;
2. $f(x_1, \ldots, x_n) = x_i$;
3. $f(x_1, \ldots, x_n) = x_i + 1$;
4. $g_1(x_1, \ldots, x_k)$, …, $g_n(x_1, \ldots, x_k)$,$h(x_1, \ldots, x_n)$ PR $\Longrightarrow$

$$f(x_1, \ldots, x_k) = h(g_1(x_1, \ldots, x_k), \ldots, g_n(x_1, \ldots, x_k)) \text{ is PR}$$

5. $h(x_1, \ldots, x_{n+1})$ and $g(x_1, \ldots, x_{n-1})$ PR $\Longrightarrow$

$$\begin{aligned} f(x_1, \ldots, x_{n-1}, 0) &= g(x_1, \ldots, x_{n-1}) \\ f(x_1, \ldots, x_{n-1}, m+1) &= h(x_1, \ldots, x_{n-1}, m, f(x_1, \ldots, x_{n-1}, m)) \end{aligned}$$

   is PR.

Examples of PR Functions

$f_0(x, y) = y + 1$. Successor.

Examples of PR Functions

$f_0(x, y) = y + 1$. Successor.

$f_1(x, y) = x + y$

Examples of PR Functions

$f_0(x, y) = y + 1$. Successor.

$f_1(x, y) = x + y$
$f_1(x, 0) = x$
$f_1(x, y + 1) = f_1(x, y) + 1$.
Used Rec Rule Once. Addition.

Examples of PR Functions

$f_0(x, y) = y + 1$. Successor.

$f_1(x, y) = x + y$
$f_1(x, 0) = x$
$f_1(x, y + 1) = f_1(x, y) + 1$.
Used Rec Rule Once. Addition.

$f_2(x, y) = xy$:

Examples of PR Functions

$f_0(x, y) = y + 1$. Successor.

$f_1(x, y) = x + y$
$f_1(x, 0) = x$
$f_1(x, y + 1) = f_1(x, y) + 1$.
Used Rec Rule Once. Addition.

$f_2(x, y) = xy$:
$f_2(x, 1) = x$ (Didn't start at 0. A detail.)
$f_2(x, y + 1) = f_2(x, y) + x$.
Used Rec Rule Twice. Once to get $x + y$ PR, and once here.
Multiplication

Examples of PR Functions

$f_0(x, y) = y + 1$. Successor.

$f_1(x, y) = x + y$
$f_1(x, 0) = x$
$f_1(x, y + 1) = f_1(x, y) + 1$.
Used Rec Rule Once. Addition.

$f_2(x, y) = xy$:
$f_2(x, 1) = x$ (Didn't start at 0. A detail.)
$f_2(x, y + 1) = f_2(x, y) + x$.
Used Rec Rule Twice. Once to get $x + y$ PR, and once here.
Multiplication

The PR functions can be put in a hierarchy depending on how many times the recursion rule is used to build up to the function.

More PR Functions

More PR Functions

$f_3(x, y) = x^y$:

More PR Functions

$f_3(x, y) = x^y$:
$f_3(x, 0) = 1$
$f_3(x, y+1) = f_3(x, y)x$.
Used Rec Rule three times. Exp.

More PR Functions

$f_3(x, y) = x^y$:
$f_3(x, 0) = 1$
$f_3(x, y + 1) = f_3(x, y)x$.
Used Rec Rule three times. Exp.

$f_4(x, y) = \mathrm{TOW}(x, y)$.

More PR Functions

$f_3(x, y) = x^y$:
$f_3(x, 0) = 1$
$f_3(x, y+1) = f_3(x, y)x$.
Used Rec Rule three times. Exp.

$f_4(x, y) = \mathrm{TOW}(x, y)$.
$f_4(x, 0) = 1$
$f_4(x, y+1) = f_4(x, y)^x$.
Used Rec Rule four times. TOWER.

More PR Functions

$f_3(x, y) = x^y$:
$f_3(x, 0) = 1$
$f_3(x, y + 1) = f_3(x, y)x$.
Used Rec Rule three times. Exp.

$f_4(x, y) = \mathrm{TOW}(x, y)$.
$f_4(x, 0) = 1$
$f_4(x, y + 1) = f_4(x, y)^x$.
Used Rec Rule four times. TOWER.

$f_5(x, y) =$ WHAT SHOULD WE CALL THIS?

More PR Functions

$f_3(x,y) = x^y$:
$f_3(x,0) = 1$
$f_3(x,y+1) = f_3(x,y)x$.
Used Rec Rule three times. Exp.

$f_4(x,y) = \mathrm{TOW}(x,y)$.
$f_4(x,0) = 1$
$f_4(x,y+1) = f_4(x,y)^x$.
Used Rec Rule four times. TOWER.

$f_5(x,y) =$ WHAT SHOULD WE CALL THIS?
$f_5(x,0) = 1$
$f_5(x,y+1) = \mathrm{TOW}(f_5(x,y),x)$.
Used Rec Rule five times.
What should we call this? Discuss

More PR Functions

$f_3(x, y) = x^y$:
$f_3(x, 0) = 1$
$f_3(x, y + 1) = f_3(x, y)x$.
Used Rec Rule three times. Exp.

$f_4(x, y) = \mathrm{TOW}(x, y)$.
$f_4(x, 0) = 1$
$f_4(x, y + 1) = f_4(x, y)^x$.
Used Rec Rule four times. TOWER.

$f_5(x, y) =$ WHAT SHOULD WE CALL THIS?
$f_5(x, 0) = 1$
$f_5(x, y + 1) = \mathrm{TOW}(f_5(x, y), x)$.
Used Rec Rule five times.
What should we call this? Discuss
Its been called WOWER.

The Functions That Have No Name

$f_a(x, y)$ is defined as

The Functions That Have No Name

$f_a(x, y)$ is defined as

$f_a(x, 0) = 1$
$f_a(x, y + 1) = f_{a-1}(f_a(x, y), x, y)$

The Functions That Have No Name

$f_a(x, y)$ is defined as

$f_a(x, 0) = 1$
$f_a(x, y + 1) = f_{a-1}(f_a(x, y), x, y)$
f_0 is Successor

The Functions That Have No Name

$f_a(x, y)$ is defined as

$f_a(x, 0) = 1$
$f_a(x, y + 1) = f_{a-1}(f_a(x, y), x, y)$
f_0 is Successor
f_1 is Addition

The Functions That Have No Name

$f_a(x, y)$ is defined as

$f_a(x, 0) = 1$
$f_a(x, y + 1) = f_{a-1}(f_a(x, y), x, y)$
f_0 is Successor
f_1 is Addition
f_2 is Multiplication

The Functions That Have No Name

$f_a(x, y)$ is defined as

$f_a(x, 0) = 1$
$f_a(x, y + 1) = f_{a-1}(f_a(x, y), x, y)$
f_0 is Successor
f_1 is Addition
f_2 is Multiplication
f_3 is Exp

The Functions That Have No Name

$f_a(x, y)$ is defined as

$f_a(x, 0) = 1$
$f_a(x, y + 1) = f_{a-1}(f_a(x, y), x, y)$
f_0 is Successor
f_1 is Addition
f_2 is Multiplication
f_3 is Exp
f_4 is Tower (This name has become standard.)

The Functions That Have No Name

$f_a(x, y)$ is defined as

$f_a(x, 0) = 1$
$f_a(x, y+1) = f_{a-1}(f_a(x, y), x, y)$
f_0 is Successor
f_1 is Addition
f_2 is Multiplication
f_3 is Exp
f_4 is Tower (This name has become standard.)
f_5 is Wower (This name is not standard.)

The Functions That Have No Name

$f_a(x, y)$ is defined as

$f_a(x, 0) = 1$
$f_a(x, y + 1) = f_{a-1}(f_a(x, y), x, y)$
f_0 is Successor
f_1 is Addition
f_2 is Multiplication
f_3 is Exp
f_4 is Tower (This name has become standard.)
f_5 is Wower (This name is not standard.)
f_6 and beyond have no name.

Levels

Def PR_a is the set of PR functions that can be defined with $\leq a$ uses of the Recursion rule.

Levels

Def PR_a is the set of PR functions that can be defined with $\leq a$ uses of the Recursion rule.

Note One can show that any finite number of exponentials is in PR_3.

More is PR than you Think

The following are PR:

More is PR than you Think

The following are PR:

1. $f(x, y) = x - y$ if $x \geq y$, 0 otherwise.

More is PR than you Think

The following are PR:

1. $f(x, y) = x - y$ if $x \geq y$, 0 otherwise.
2. $f(x, y) =$ the quotient when you divide x by y.

More is PR than you Think

The following are PR:

1. $f(x,y) = x - y$ if $x \geq y$, 0 otherwise.
2. $f(x,y) =$ the quotient when you divide x by y.
3. $f(x,y) =$ the remainder when you divide x by y.

More is PR than you Think

The following are PR:

1. $f(x, y) = x - y$ if $x \geq y$, 0 otherwise.
2. $f(x, y) =$ the quotient when you divide x by y.
3. $f(x, y) =$ the remainder when you divide x by y.
4. $f(x, y) = x \pmod{y}$.

More is PR than you Think

The following are PR:

1. $f(x, y) = x - y$ if $x \geq y$, 0 otherwise.
2. $f(x, y) =$ the quotient when you divide x by y.
3. $f(x, y) =$ the remainder when you divide x by y.
4. $f(x, y) = x \pmod{y}$.
5. $f(x, y) = \mathrm{GCD}(x, y)$.

More is PR than you Think

The following are PR:

1. $f(x, y) = x - y$ if $x \geq y$, 0 otherwise.
2. $f(x, y) =$ the quotient when you divide x by y.
3. $f(x, y) =$ the remainder when you divide x by y.
4. $f(x, y) = x \pmod{y}$.
5. $f(x, y) = \mathrm{GCD}(x, y)$.
6. $f(x) = 1$ if x is prime, 0 if not.

More is PR than you Think

The following are PR:

1. $f(x, y) = x - y$ if $x \geq y$, 0 otherwise.
2. $f(x, y) =$ the quotient when you divide x by y.
3. $f(x, y) =$ the remainder when you divide x by y.
4. $f(x, y) = x \pmod{y}$.
5. $f(x, y) = \mathrm{GCD}(x, y)$.
6. $f(x) = 1$ if x is prime, 0 if not.
7. $f(x) = 1$ if x is the sum of 2 primes, 0 otherwise.

A Natural non PR Function

Def **Ackermann's function** is the function defined by

$$\begin{aligned} A(0, y) &= y + 1 \\ A(x+1, 0) &= A(x, 1) \\ A(x+1, y+1) &= A(x, A(x+1, y)) \end{aligned}$$

A Natural non PR Function

Def **Ackermann's function** is the function defined by

$$\begin{aligned} A(0, y) &= y + 1 \\ A(x+1, 0) &= A(x, 1) \\ A(x+1, y+1) &= A(x, A(x+1, y)) \end{aligned}$$

1. A is obviously computable.

A Natural non PR Function

Def **Ackermann's function** is the function defined by

$$\begin{aligned} A(0,y) &= y+1 \\ A(x+1,0) &= A(x,1) \\ A(x+1,y+1) &= A(x,A(x+1,y)) \end{aligned}$$

1. A is obviously computable.
2. A grows faster than any PR function.

A Natural non PR Function

Def **Ackermann's function** is the function defined by

$$\begin{aligned} A(0,y) &= y+1 \\ A(x+1,0) &= A(x,1) \\ A(x+1,y+1) &= A(x,A(x+1,y)) \end{aligned}$$

1. A is obviously computable.
2. A grows faster than any PR function.
3. Since A is defined using a recursion which involves applying the function to itself there is no obvious way to take the definition and make it PR. Not a proof, an intuition.

Ackermann's Function is Natural: Security

https://www.ackermansecurity.com/blog/home-security-tips/5-ways-home-security-signs-prevent-burglaries

Ackermann's Function is Natural: Security

https://www.ackermansecurity.com/blog/home-security-tips/5-ways-home-security-signs-prevent-burglaries

They are called Ackerman Security since they claim that Burglar would have to be Ackerman(n)-good to break in.

Ackermann's Function is Natural: DS

DS is Data Structure.
UNION-FIND DS for sets that supports:

Ackermann's Function is Natural: DS

DS is Data Structure.
UNION-FIND DS for sets that supports:
(1) If a is a number then make $\{a\}$ a set.

Ackermann's Function is Natural: DS

DS is Data Structure.
UNION-FIND DS for sets that supports:
(1) If a is a number then make $\{a\}$ a set.
(2) If A, B are sets then make $A \cup B$ a set.

Ackermann's Function is Natural: DS

DS is Data Structure.
UNION-FIND DS for sets that supports:
(1) If a is a number then make $\{a\}$ a set.
(2) If A, B are sets then make $A \cup B$ a set.
(3) Given x find which, if any, set it is in.

Ackermann's Function is Natural: DS

DS is Data Structure.
UNION-FIND DS for sets that supports:
(1) If a is a number then make $\{a\}$ a set.
(2) If A, B are sets then make $A \cup B$ a set.
(3) Given x find which, if any, set it is in.

- There is a DS for this problem that can do n operations in $nA^{-1}(n)$ steps.
- One can show that there is no better DS.

So $nA^{-1}(n, n)$ is the exact upper and lower bound!

Ackermann's Function and Goodstein Seq

Writing a number as a sum of powers of 2.

$$1000 = 2^9 + 2^8 + 2^7 + 2^6 + 2^5 + 2^3$$

Ackermann's Function and Goodstein Seq

Writing a number as a sum of powers of 2.

$$1000 = 2^9 + 2^8 + 2^7 + 2^6 + 2^5 + 2^3$$

But we can also write the exponents as sums of power of 2

$$1000 = 2^{2^3+2^0} + 2^{2^3} + 2^{2^2+2^1+2^0} + 2^{2^1+2^0}$$

Ackermann's Function and Goodstein Seq

Writing a number as a sum of powers of 2.

$$1000 = 2^9 + 2^8 + 2^7 + 2^6 + 2^5 + 2^3$$

But we can also write the exponents as sums of power of 2

$$1000 = 2^{2^3+2^0} + 2^{2^3} + 2^{2^2+2^1+2^0} + 2^{2^1+2^0}$$

We can even write the exponents that are not already powers of 2 as sums of powers of 2.

$$1000 = 2^{2^{2^1+2^0}+2^0} + 2^{2^{2^1+2^0}} + 2^{2^2+2^1+2^0} + 2^{2^1+2^0}$$

Ackermann's Function and Goodstein Seq

Writing a number as a sum of powers of 2.

$$1000 = 2^9 + 2^8 + 2^7 + 2^6 + 2^5 + 2^3$$

But we can also write the exponents as sums of power of 2

$$1000 = 2^{2^3+2^0} + 2^{2^3} + 2^{2^2+2^1+2^0} + 2^{2^1+2^0}$$

We can even write the exponents that are not already powers of 2 as sums of powers of 2.

$$1000 = 2^{2^{2^1+2^0}+2^0} + 2^{2^{2^1+2^0}} + 2^{2^2+2^1+2^0} + 2^{2^1+2^0}$$

This is called **Hereditary Base n Notation**

Ackermann's Function and Goodstein Seq

$$1000 = 2^{2^{2^1+2^0}+2^0} + 2^{2^{2^1+2^0}} + 2^{2^2+2^1+2^0} + 2^{2^1+2^0}$$

Replace all of the 2's with 3's:

$$3^{3^{3^1+3^0}+3^0} + 3^{3^{3^1+3^0}} + 3^{3^3+3^1+3^0} + 3^{3^1+3^0}$$

Ackermann's Function and Goodstein Seq

$$1000 = 2^{2^{2^1+2^0}+2^0} + 2^{2^{2^1+2^0}} + 2^{2^2+2^1+2^0} + 2^{2^1+2^0}$$

Replace all of the 2's with 3's:

$$3^{3^{3^1+3^0}+3^0} + 3^{3^{3^1+3^0}} + 3^{3^3+3^1+3^0} + 3^{3^1+3^0}$$

This number just went WAY up. Now subtract 1.

$$3^{3^{3^1+3^0}+3^0} + 3^{3^{3^1+3^0}} + 3^{3^3+3^1+3^0} + 3^{3^1+3^0} - 1$$

Ackermann's Function and Goodstein Seq

$$1000 = 2^{2^{2^1+2^0}+2^0} + 2^{2^{2^1+2^0}} + 2^{2^2+2^1+2^0} + 2^{2^1+2^0}$$

Replace all of the 2's with 3's:

$$3^{3^{3^1+3^0}+3^0} + 3^{3^{3^1+3^0}} + 3^{3^3+3^1+3^0} + 3^{3^1+3^0}$$

This number just went WAY up. Now subtract 1.

$$3^{3^{3^1+3^0}+3^0} + 3^{3^{3^1+3^0}} + 3^{3^3+3^1+3^0} + 3^{3^1+3^0} - 1$$

Repeat the process:
Replace 3 by 4, and subtract 1, Replace 4 by 5, and subtract 1, $\cdots$.

Ackermann's Function and Goodstein Seq

$$1000 = 2^{2^{2^1+2^0}+2^0} + 2^{2^{2^1+2^0}} + 2^{2^2+2^1+2^0} + 2^{2^1+2^0}$$

Replace all of the 2's with 3's:

$$3^{3^{3^1+3^0}+3^0} + 3^{3^{3^1+3^0}} + 3^{3^3+3^1+3^0} + 3^{3^1+3^0}$$

This number just went WAY up. Now subtract 1.

$$3^{3^{3^1+3^0}+3^0} + 3^{3^{3^1+3^0}} + 3^{3^3+3^1+3^0} + 3^{3^1+3^0} - 1$$

Repeat the process:
Replace 3 by 4, and subtract 1, Replace 4 by 5, and subtract 1, $\cdots$.
Vote Does the sequence:

- Goto infinity (and if so how fast- perhaps Ack-like?)
- Eventually stabilizes (e.g., goes to 18 and then stops there)
- Cycles- goes UP then DOWN then UP then DOWN ....

Ackermann's Function and Goodstein Seq

$$1000 = 2^{2^{2^1+2^0}+2^0} + 2^{2^{2^1+2^0}} + 2^{2^2+2^1+2^0} + 2^{2^1+2^0}$$

Replace all of the 2's with 3's:

$$3^{3^{3^1+3^0}+3^0} + 3^{3^{3^1+3^0}} + 3^{3^3+3^1+3^0} + 3^{3^1+3^0}$$

This number just went WAY up. Now subtract 1.

$$3^{3^{3^1+3^0}+3^0} + 3^{3^{3^1+3^0}} + 3^{3^3+3^1+3^0} + 3^{3^1+3^0} - 1$$

Repeat the process:
Replace 3 by 4, and subtract 1, Replace 4 by 5, and subtract 1, $\cdots$.
Vote Does the sequence:

- Goto infinity (and if so how fast- perhaps Ack-like?)
- Eventually stabilizes (e.g., goes to 18 and then stops there)
- Cycles- goes UP then DOWN then UP then DOWN ....

The sequence goes to 0.
The number of steps for n to goto 0 is roughly $ACK(n, n)$.

Upshot

Upshot

1. The PR functions give us levels to measure how fast functions grow.

Upshot

1. The PR functions give us levels to measure how fast functions grow.
2. Almost all functions from N^k to N encountered in mathematics are PR.

Upshot

1. The PR functions give us levels to measure how fast functions grow.
2. Almost all functions from N^k to N encountered in mathematics are PR.
3. Ackermann's function is computable and not PR.

Upshot

1. The PR functions give us levels to measure how fast functions grow.
2. Almost all functions from N^k to N encountered in mathematics are PR.
3. Ackermann's function is computable and not PR.
4. Ackerman's function grows faster than any PR function.

Upshot

1. The PR functions give us levels to measure how fast functions grow.
2. Almost all functions from $\mathbb{N}^k$ to $\mathbb{N}$ encountered in mathematics are PR.
3. Ackermann's function is computable and not PR.
4. Ackerman's function grows faster than any PR function.
5. If we want to indicate that a function grows **really fast** we may compare it to Ackermann's function.