A DECIDABLE THEORY: $(\mathbb{Q}, <)$

Variables and Symbols for $(\mathbb{Q}, <)$

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Consider the following language.

1. The logical symbols $\wedge$, $\neg$, $(\exists)$.
2. Variables $x, y, z, \ldots$ that range over $\mathbb{Q}$.
3. Constants: all elements of $\mathbb{Q}$.
4. The symbols $<$ and $=$. **Note** We do not have $+$ or $\times$.

Atomic Formulas

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   2.1 $\phi_1 \wedge \phi_2$,
   2.2 $\phi_1 \vee \phi_2$
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3. If $\phi(x_1, \ldots, x_n)$ is a QL Formula then so is $(\exists x_i)[\phi(x_1, \ldots, x_n)]$

The Theory of $(\mathbb{Q}, <)$

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The following problem is decidable.

- Input ϕ, a sentence in $(\mathbb{Q}, <)$.
- Determine if ϕ is TRUE.

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which is equiv to

$$(\exists w)(\forall x[(\exists y)[w < x < y] \vee (\exists y)[w < y < x] \vee (\exists y)[w < y = x]]$$

Can then look at each piece separately.

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So

$$(\exists w)(\forall x)[(\exists y)[w < x < y] \vee (\exists y)[w < y < x] \vee (\exists y)[w < y = x]] \equiv$$

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So

$$(\exists w)(\forall x)[(\exists y)[w < x < y] \vee (\exists y)[w < y < x] \vee (\exists y)[w < y = x]] \equiv$$

$$(\exists w)(\forall x)[(\exists y)[w < x] \vee (\exists y)[w < x] \vee (\exists y)[w < x]] \equiv$$

$$(\exists w)(\forall x)[(w < x) \vee (w < x) \vee (w < x))] \equiv (\exists w)(\forall x)[w < x]$$

Key We elim a $\exists y$! That elim clauses is incidental.

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But recall that $\forall \equiv \neg\exists\neg$. **Yeah**

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$$(\exists w)\neg(\exists x)[x \leq w]$$

Look at the inner part:

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Look at the inner part:

$$(\exists x)[x \le w] \equiv \mathrm{TRUE}$$

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Look at the inner part:

$$(\exists w)\neg(\exists x)[x \le w] \equiv (\exists w)[\neg\mathrm{TRUE}] \equiv$$

$$(\exists w)[\mathrm{FALSE}] \equiv \mathrm{FALSE}$$

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Look at the inner part:

$$(\exists w)\neg(\exists x)[x \le w] \equiv (\exists w)[\neg\mathrm{TRUE}] \equiv$$

$$(\exists w)[\mathrm{FALSE}] \equiv \mathrm{FALSE}$$

So the original statement is FALSE.

Lemma on Quantifier Elimination

Lemma $\exists$ an algorithm that will, given a sentence of the form

$$(Q_1 x_1) \cdots (Q_{n-1} x_{n-1})(\exists x_n)[\phi(x_1, \ldots, x_n)]$$

(where the Q_i are quantifiers) return a sentence of the form

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Replace $\phi(x_1,\ldots,x_n)$ with an OR of all poss. orderings of $x_1,\ldots,x_n$.

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Replace $\phi(x_1, \ldots, x_n)$ with an OR of all poss. orderings of $x_1, \ldots, x_n$.
Then replace

$(\exists x_n)[L_1(x_1, \ldots, x_n) \vee \cdots \vee L_m(x_1, \ldots, x_n)]$ with
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Each part is either $\equiv$ to the part with x_n removed OR T or F.

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$(\mathbb{Q}, <)$ is Decidable: The Algorithm

Algorithm

1. $(Q_1 x_1) \cdots (Q_n x_n)[\phi(x_1, \ldots, x_n)]$. Replace $\forall$ with $\neg\exists\neg$.
2. Apply the Quant Elim Lemma over and over again until either you end up with a TRUE or a FALSE or a sentence with one variable whose truth will be easily discerned (see next slide for more on that).

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$(\mathbb{Q}, <)$ is decidable! **Great!** We can take all of the open questions about $(\mathbb{Q}, <)$ and use the decision procedure to solve them!

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1. How many ways you order $x_1, \ldots, x_n$. We all know this is $n!$.
2. How many ways you order $x_1, \ldots, x_n$ if you allow $=$? Next slide for examples and the first few numbers.

The Horse Numbers and $H(3)$

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There are 3 where x_1 is unique least.

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There are 3 where x_3 is unique least.

If $x_1 = x_2$ is least: $x_1 = x_2 < x_3$. There is 1.

The Horse Numbers and $H(3)$

$H(n)$ is the number of ways that n horses can finish a race. Note that some could be tied.

$H(2) = 3$: $x_1 < x_2$, $x_2 < x_1$, $x_1 = x_2$.

$H(3)$ we will derive. If x_1 is unique least:

$x_1 < x_2 < x_3$

$x_1 < x_2 = x_3$

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If $x_1 = x_2 = x_3$ there is 1.

Total $H(3) = 3 + 3 + 3 + 1 + 1 + 1 + 1 = 13$.

The Horse Numbers: $H(4)$

$$H(1) = 1 \qquad H(2) = 3 \qquad H(3) = 13.$$

The Horse Numbers: $H(4)$

$H(1) = 1 \qquad H(2) = 3 \qquad H(3) = 13.$

Work with your neighbor to try to derive $H(4)$.

Hint: You use $H(2)$ and $H(3)$.

The Horse Numbers: $H(4)$

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The Horse Numbers: $H(4)$

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1. There is ONE min. $\binom{4}{1} \times H(3)$.

The Horse Numbers: $H(4)$

$$H(0) = 1 \qquad H(1) = 1 \qquad H(2) = 3 \qquad H(3) = 13.$$

1. There is ONE min. $\binom{4}{1} \times H(3)$.
2. There are TWO mins. $\binom{4}{2} \times H(2)$.

The Horse Numbers: $H(4)$

$$H(0) = 1 \qquad H(1) = 1 \qquad H(2) = 3 \qquad H(3) = 13.$$

1. There is ONE min. $\binom{4}{1} \times H(3)$.
2. There are TWO mins. $\binom{4}{2} \times H(2)$.
3. There are THREE mins. $\binom{4}{3} \times H(1)$.

The Horse Numbers: $H(4)$

$$H(0) = 1 \qquad H(1) = 1 \qquad H(2) = 3 \qquad H(3) = 13.$$

1. There is ONE min. $\binom{4}{1} \times H(3)$.
2. There are TWO mins. $\binom{4}{2} \times H(2)$.
3. There are THREE mins. $\binom{4}{3} \times H(1)$.
4. There are FOUR mins. $\binom{4}{4} \times H(0)$.

The Horse Numbers: $H(4)$

$$H(0) = 1 \qquad H(1) = 1 \qquad H(2) = 3 \qquad H(3) = 13.$$

1. There is ONE min. $\binom{4}{1} \times H(3)$.
2. There are TWO mins. $\binom{4}{2} \times H(2)$.
3. There are THREE mins. $\binom{4}{3} \times H(1)$.
4. There are FOUR mins. $\binom{4}{4} \times H(0)$.

Total

$$H(4) = \binom{4}{1} \times H(3) + \binom{4}{2} \times H(2) + \binom{4}{3} \times H(1) + \binom{4}{0} \times H(0) = 75.$$

The Horse Numbers: Recurrence

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$H(n)$:

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$\vdots$)

$n-1$) There are $n-1$ mins. $\binom{n}{n-1} \times H(1)$.

The Horse Numbers: Recurrence

$H(n)$:

1) There is ONE min. $\binom{n}{1} \times H(n-1)$.
2) There are TWO mins. $\binom{n}{2} \times H(n-2)$.

$\vdots$)

$n-1$) There are $n-1$ mins. $\binom{n}{n-1} \times H(1)$.

n) There are n mins. $\binom{n}{n} \times H(0)$.

$$H(n) = \binom{n}{1} H(n-1) + \cdots + \binom{n}{n} H(0).$$

The Bill Numbers

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$B(2) = 1$

The Bill Numbers

$B(n)$ is the number of ways n horses can finish GIVEN that $x_1 < x_2$.

$B(2) = 1$

$B(3) = 5$.

$x_1 < x_2 < x_3$

$x_1 < x_2 = x_3$

$x_1 < x_3 < x_2$

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The Bill Numbers

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$B(2) = 1$

$B(3) = 5$.

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There may be a HW where you find $B(4)$ and get a recurrence for $B(n)$. (The recurrence will also use the H numbers.)