HW 2 CMSC 456. Morally DUE Sep 16
SOLUTIONS
NOTE- THE HW IS SIX PAGES LONG

1. (0 points) READ the syllabus- Content and Policy. READ my NOTES on ciphers and on English. What is your name? What is the day and time of the midterm?

2. (15 points) Klingons use an alphabet of 35 letters. Vulcans use an alphabet of 36 letters. Romulans use an alphabet of 37 letters. Spock notes that Vulcans have an easier time using the Playfair cipher than Klingons or Romulans. He is correct.

   (a) (5 points) Why is it easier for Vulcans to use the Playfair cipher than Klingons or Romulans?

   (b) (5 points) For Klingons to use the Playfair cipher what do they need to do initially?

   (c) (5 points) For Romulans to use the Playfair cipher what do they need to do initially?

SOLUTION TO PROBLEM TWO

(a) Why is it easier for Vulcans to use the PLAYFAIR cipher than Klingons or Romulans? ANSWER: Since for Vulcans the number of letters is already a square they do not need to fiddle with the letters.

(b) For Klingons to use the Playfair cipher what do they need to do initially? ANSWER: Klingons need to add one dummy character to their alphabet so it has 36, a square.

(c) For Romulans to use the Playfair cipher what do they need to do initially? ANSWER: Romulans need to select two characters to merge together into 1, perhaps the 2 least used characters, so that their alphabet will have 36.

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3. (20 points) Alice and Bob are using the Vig Cipher. They RANDOMLY generate a 3-letter word and a 4-letter word and then use that trick in class (Sept 4 lecture) to get a 12-letter word out of it.

   (a) (5 points) The key phrase generated is *bbb cccc*. What is the new 12-letter key phrase?

   (b) (5 points) Encode the word *Nathan* with the key phrase obtained from part (a) using the Vig cipher.

   (c) (5 points) Is there another name for the cipher you are using?

   (d) (5 points) You may have noticed that this is not as strong as Vig is supposed to be. Is there something Alice and Bob could do to avoid this kind of case?

   (e) (0 points) Should Alice and Bob take your advice?

SOLUTION TO PROBLEM THREE

   (a) The key phrase is initially *bbb cccc*. What is the new longer keyword? ANSWER: $b = 1$ and $c = 2$ so $b + c = 3 = d$. The new word is

   $(b + c) \cdots (b + c)$ 12 times, so *ddddddddddd*.

   (b) Encode the word *Nathan* with the key using Vig cipher. ANSWER: shift every letter by $d = 3$. We omit.

   (c) Is there another name for the cipher you are using? ANSWER: Yes. It's SHIFT by 3. This is also called *the Caesar Cipher*. Wikipedia claims that Caesar used this with a shift of 3 so this *really is* the Caesar Cipher. (I have read that Ceaser used shift-3 in other sources; however, I am skeptical that we know such things.)

   (d) You may have noticed that this is not as strong as Vig is supposed to be. Is there something Alice and Bob could do to avoid this kind of case. ANSWER: When you generate a random string, if they are ALL the same letter than generate another random string.

   (e) Should Alice and Bob take your advice? ANSWER: This is a matter of opinion but I would say NO. You are cutting down your search space. And just because Alice and Bob know that its just a Shift +3 cipher, Eve does not know that.

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4. (30 points) This is a programming problem. You will write a program that performs the following tasks, outputting 52 lines in total.

   (a) Input (from standard input) a string of (English) text. For this assignment, you will be processing all letters that appear in the text. The string of text input may contain non-alphabetic characters (e.g., punctuation, whitespace, etc.), so you should discard / ignore any such characters you encounter. You may assume that all the text is contained in a single line of input. We consider lowercase and uppercase letters to be equivalent.

   (b) Recall how we associate each letter with a corresponding number (e.g., $a \mapsto 0, b \mapsto 1, \ldots, z \mapsto 25$). For each $0 \leq c \leq 25$, compute $B[c]$, which denotes the proportion of times c appeared in the text. For instance, if the string has 40 letters and c appears 17 times, then $B[c] = 17/40 = 0.425$.

   Once you have done this, go through each $0 \leq c \leq 25$ (starting with $c = 0$ and ending with $c = 25$) and print (to standard output) $B[c]$ on its own line. (So you should print 26 lines for this part.) Make sure you are printing each $B[c]$ as a decimal (i.e., not as a fraction).

   (c) For each $0 \leq s \leq 25$, compute array C_s, the circular shift of array B from part (b) by shift s (i.e., for each $0 \leq k \leq 25$, $C_s[k] = B[k + s \pmod{26}]$). Now, go through each $0 \leq s \leq 25$ (starting with $s = 0$ and ending with $s = 25$) and print the dot product of C_s and B on its own line. (Recall that the dot product of two vectors u, v of length n is defined as $u \cdot v = \sum_{k=1}^{n} u_k v_k$.)

   Note: We expect to find that shifting by 0 results in a dot product of roughly 0.065 and shifting by anything else results in a value of roughly ≤ 0.045. If you do not get this then recheck your work, but it may still be correct if the input text is unusual in some way.

YOU ARE NOT DONE! GOTO NEXT PAGE TO SEE WHAT YOU RUN THIS PROGRAM ON

Run your program on the text from

`https://pastebin.com/raw/wwUeULYU`

What was the largest dot product value you obtained form part (c)? What was the second largest? Report these values along with the rest of your homework. Your code should be uploaded separately (see below).

You are free to choose from various programming languages to complete this problem. By default, we support C, C++, Java, Python2/3, and Ruby. Ask on Piazza if you want more options. You will be submitting all code files you used to complete this problem to the Gradescope assignment called "hw02 - problem 4". Since you will probably want to submit multiple files, you should merge all files into a single zip file and submit that zip file to Gradescope. Upon submission, your code will be automatically run on a Linux machine and tested against various test cases to ensure correctness. You are allowed to submit your code as many times as you want.

Regardless of the language you choose, your submission must include a bash script called `run` (with no file extension). This file must begin with the shebang `#!/usr/bin/env bash` on the very first line. This script will be run each time the autograder tries to run your code, so add to this file any commands that are needed to run your code. This gives you greater flexibility regarding how you want to organize your code. Additionally, if you are using a non-scripting language such as Java, also upload a bash script called `build`, also with shebang. This script will be called once upon submission to compile your code before execution.

If you have any questions or confusions, or if you encounter any technical difficulties, feel free to ask for help on Piazza.

SOLUTION TO PROBLEM FOUR

Omitted.

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5. (20 points) Let

$$M = 2^2 \times 3^3 \times 11 \times 197$$

(a) (5 points) How many positive factors does M have? (Include 1 and M itself.)

(b) (15 points) Find all positive factors of M that are between 7000 and 9999.

Do it by hand and show your work. It is NOT enough to show one such number, you must show all such numbers and prove there are no other ones. Also, DO NOT check all factors using brute force; you should use reasoning to reduce your search space.

(NOTE- these numbers are different from the ones on the slides.)

SOLUTION TO PROBLEM FIVE

(a) How many positive factors does M have? (Include 1 and M itself.)
ANSWER:

$$M = 2^2 \times 3^3 \times 11 \times 197$$

Hence all of the factors of M are of the form

$$M = 2^a \times 3^b \times 11^c \times 197^d$$

where
$(a, b, c, d) \in \{0, 1, 2\} \times \{0, 1, 2, 3\} \times \{0, 1\} \times \{0, 1\}$.
Hence the number of factors is $3 \times 4 \times 2 \times 2 = 48$.

(b) Find all factors of

$$2^2 \times 3^3 \times 11 \times 197$$

that are between 7000 and 9999.

Do it by hand and show your work.

(NOTE- these numbers are different from the ones on the slides.)
ANSWER:

- *Case 1:* The factor does not have 197. Then the factor is $\leq 2^2 \times 3^3 \times 11 = 1188 < 7000$. All future cases assume the factor has 197.
- *Case 2:* The factor does not have 11. Then the factor has zero, one, or two 2's and zero, one, two, or three 3's.
  i. If it has at most one 3 then the factor is $\leq 2^2 \times 3 \times 197 = 2364 < 7000$. (All future subcases of Case 2 assume two or three 3's.)
  ii. If it has two 3's then the factor is $2^k \times 3^2 \times 197 = 2^k \times 1773$ for some $0 \leq k \leq 2$. $k = 2$ yields 7092 - YES! that works! $k = 0$ and $k = 1$ do not work. (We leave it to the reader to check that in both cases the number is too small. In fact, just checking that the $k = 1$ case is too small will suffice.)
  iii. If it has three 3's then the factor is $2^k \times 3^3 \times 197 = 2^k \times 5319$ for some $0 \leq k \leq 2$. No k works in this case. (We leave it to the reader to check that if $k = 0$ then the number is too small, and if $k = 1$ or $k = 2$ then the number is too large.)

  All future cases assume the factor has 11 and 197.
- *Case 3:* The factor has two or three 3's. Then the factor is $\geq 197 \times 11 \times 9 = 19503 > 9999$. All future cases assume that the factor has 11 and 197 but at most one 3.
- *Case 4:* The factor has one 3. Then the factor has either zero, one, or two 2's.
  - If the factor has zero 2's then the factor is $3 \times 11 \times 197 = 6501 < 7000$.
  - If the factor has one or two 2's then the factor is $2 \times 3 \times 11 \times 197 = 13002 > 9999$.
- *Case 4:* The factor has zero 3's. Then the factor has either zero, one, or two 2's.
  - If the factor has zero or one 2's then the factor is $\leq 2 \times 11 \times 197 = 4334 < 7000$.
  - If the factor has two 2's then the factor is $4 \times 11 \times 197 = 8668$. YES! THIS IS IN THE RANGE NEEDED!

Hence the only such factors are 7092 and 8668.

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6. (15 points) (Please do by hand – the numbers do not get that big. I cannot stop you from using a computer; however, you will get more out of the exercise if you do it by hand.)

   (a) (5 points) Which numbers in $\{1, 2, 3, \ldots, 17\}$ have an inverse mod 18?

   (b) (5 points) For all such numbers, give the inverse.

   (c) (5 points) Show that, for all n, the inverse of $n - 1 \pmod{n}$ is actually $n - 1$.

SOLUTION TO PROBLEM SIX

a) The numbers are 1,5,7,11,13,17.

b) I also tell you my thought process.

We are always on the lookout for 19, 37, 55, 73, 91 which are all 1 mod 18.

1 has inverse 1

5 has inverse 11 since $5 \times 11 = 55 \equiv 1 \pmod{18}$

7 has inverse 13 since $7 \times 13 = 91 \equiv 1 \pmod{15}$

11 has inverse 5

13 has inverse 7

17 has inverse 17

c) $(n-1)^2 = n^2 - 2n + 1 \equiv 1 \pmod{n}$.