Public Key Cryptography: RSA

September 30, 2019
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The Academic Code, More Examples

When Academics Says: The agreement of my theory and the empirical data is Excellent.

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Recall If $p$ prime, $a \not\equiv 0 \pmod{p}$, then $a^{p-1} \equiv 1 \pmod{p}$.

How to compute $3^{1000} \pmod{7}$?

Could do repeated squaring. Can we do better? Discuss.
Exponentiation Mod $p$ Revisited

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so

$$3^{1000} \equiv 3^{6 \times 166 + 4} \equiv (3^6)^{166} \times 3^4 \equiv 3^4$$
Exponentiation Mod $p$ Revisited: Another Example

$11^{999,999,999} \pmod{107}$

Repeated squaring would take at least $\lg(999,999,999) \sim 30 \times$'s.
Exponentiation Mod $p$ Revisited: Another Example

$11^{999,999,999}$ (mod 107)

Repeated squaring would take at least $\log(999,999,999) \sim 30 \times$’s.
By Fermat's Little Theorem $11^{106} \equiv 1$ (mod 107).
Divide 999,999,999 by 106:

$999,999,999 = 106k + 27$ (don’t care what $k$ is)
Exponentiation Mod \( p \) Revisited: Another Example

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Lets rewrite that

$11^{999,999,999} \equiv 11^{999,999,999} \pmod{106} \pmod{107} \equiv 11^{27} \pmod{107}$

Now do normal repeated squaring. 10 $\times$’s total.
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Now do normal repeated squaring. 10 $\times$'s total.

Can we generalize? Yes
Generalize $p$ prime, $a \not\equiv 0 \pmod{p}$, $m \in \mathbb{N}$.

We want to compute $a^m$.

We know that $a^{p-1} \equiv 1 \pmod{p}$.

Divide $m$ by $p - 1$: $m = k(p - 1) + r$ where $0 \leq r \leq p - 2$ and $r \equiv m \pmod{p - 1}$.

Hence:

$$a^m \equiv a^{k(p-1)+r} \equiv (a^{p-1})^k \times a^r \equiv 1^k a^r \equiv a^r$$

But recall that $r \equiv m \pmod{p - 1}$. So

$$a^m \equiv a^{m \mod p-1} \pmod{p}$$

This last equation is the important point.
Next few slides are on the $\phi$ function.

YES, you have already seen it.

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Math is best learned twice... at least twice.
Needed Mathematics - The $\phi$ Function

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Needed Mathematics- The \( \phi \) Function

Recall If \( p \) is prime and \( 1 \leq a \leq p - 1 \) then \( a^{p-1} \equiv 1 \) (mod \( p \)).

Recall: For all \( m \), \( a^m \equiv a^m \pmod{p-1} \) (mod \( p \)).

So arithmetic in the exponents is mod \( p - 1 \).

We need to generalize this to when the mod is not a prime.

**Definition**

\( \phi(n) \) is the number of numbers in \( \{1, \ldots, n\} \) that are relatively prime to \( n \).

Recall: If \( p \) is prime then \( \phi(p) = p - 1 \).

Recall: If \( a, b \) rel prime then \( \phi(ab) = \phi(a)\phi(b) \).
Theorem for Primes, Theorem for $n$

We restate and generalize.

**Fermat’s Little Theorem:** If $p$ is prime and $a \not\equiv 0 \pmod{p}$ then

$$a^m \equiv a^{m \mod p-1} \pmod{p}.$$ 

Restate:

**Fermat’s Little Theorem:** If $p$ is prime and $a$ is rel prime to $p$ then

$$a^m \equiv a^{m \mod p-1} \pmod{p}.$$ 

Generalize:

**Fermat-Euler Theorem:** If $n \in \mathbb{N}$ and $a$ is rel prime to $n$ then

$$a^m \equiv a^{m \mod \phi(n)} \pmod{n}.$$
Examples

\[ 14^{999,999} \pmod{393} \]

\[ \phi(393) = \phi(3 \times 131) = \phi(3) \times \phi(131) = 2 \times 130 = 260. \]

\[ 14^{999,999} = 14^{999,999} \pmod{260} \pmod{393} \equiv 14^{39} \pmod{393} \]

Now just do repeated squaring.
Bait and Switch

I got you interested in the theorem

\[ a^m \equiv a^{m \mod \phi(n)} \quad \pmod{n} \]

by telling you that it can be used to do things like

\[ 17^{191,992,194,299,292777} \quad \pmod{150}. \]

with FAR less than \( 2 \lg(191, 992, 194, 299, 292777) \times 's. \)
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\[ a^m \equiv a^m \mod \phi(n) \pmod{n} \]
Known to be Easy, Do in Order

1. Given $L$, generate two primes of length $L$: $p, q$.
2. Compute $N = pq$ and $R = (p - 1)(q - 1)$.
3. Find $e$ rel prime to $R$.
4. If have $p, q$ then Find $d$ such that $ed \equiv 1 \pmod{R}$. KEY: Easy since have $p, q$. Would be hard otherwise
5. Compute $m^e \pmod{N}$.

Thought to be Hard

Given $N, e$ as above find $d$ as above. Note that we are not given $p, q$ or $R$. 

RSA

Let $L$ be a security parameter

1. Alice picks two primes $p$, $q$ of length $L$ and computes $N = pq$.

2. Alice computes $\phi(N) = \phi(pq) = (p-1)(q-1)$. Denote by $R$.

3. Alice picks an $e \in \{1, \ldots, 2R\}$ that is relatively prime to $R$.

4. Alice finds $d$ such that $ed \equiv 1 \pmod{R}$.

5. Alice broadcasts $(N, e)$. (Bob and Eve both see it.)

6. Bob: To send $m \in \{1, \ldots, N-1\}$, send $m^e (\bmod N)$.

7. If Alice gets $m^e (\bmod N)$ she computes $(m^e)^d \equiv m^{ed} \equiv m^1 \pmod{R} \equiv m$.

Note: Works $1 \leq m \leq N-1$. $m$ need not be rel prime to $N$.

PRO: Alice and Bob can execute the protocol easily.

Biggest PRO: Alice and Bob never had to meet!

Question: Can Eve find out $m$?
RSA

Let $L$ be a security parameter

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3. **Alice** picks an $e \in \{ \frac{R}{3}, \ldots, \frac{2R}{3} \}$ that is relatively prime to $R$.
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Question: Can Eve find out $m$?
Convention for RSA

Alice sends \((N, e)\) to get the process started
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Then Bob can send Alice messages.
Alice sends \((N, e)\) to get the process started

Then Bob can send Alice messages.

We don’t have Alice sending Bob messages.
Pick out two students to be Alice and Bob.

Use primes

\( p = 31 \), Prime

\( q = 37 \), Prime

\( N = pq = 31 \times 37 = 1147 \).

\( R = \phi(N) = 30 \times 36 = 1080 \)

\( e = 77 \), \( e \) rel prime to \( R \)

\( d = 533 \) \( (ed \equiv 1 \pmod{R}) \)

**CHECK:** \( ed = 77 \times 533 = 41041 \equiv 1 \pmod{1080} \).

**Bob:** pick an \( m \in \{1, \ldots, N - 1\} = \{1, \ldots, 1146\} \). Do not tell us what it is.

**Bob:** compute \( c = m^e \pmod{1147} \) and tell it to us.

**Alice:** compute \( c^d \pmod{1147} \), should get back \( m \).
What Do We Really Know about RSA

If Eve can factor then she can crack RSA.

1. Input \((N, e)\) where \(N = pq\) and \(e\) is rel prime to \(R = (p - 1)(q - 1)\). (\(p, q, R\) are NOT part of the input.)
2. Eve factors \(N\) to find \(p, q\). Eve computes \(R = (p - 1)(q - 1)\).
3. Eve finds \(d\) such that \(ed \equiv 1 \pmod{R}\).

If Factoring Easy then RSA is crackable

VOTE: TRUE or FALSE or UNKNOWN TO SCIENCE

Note: In ugrad math classes rare to have a statement that is UNKNOWN TO SCIENCE. Discuss.
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Hardness Assumption

Definition
Let $f$ be the following function:
Input: $N, e, m^e \pmod{N}$ (know $N = pq$ but don’t know $p, q$).
Outputs: $m$.

Hardness assumption (HA): $f$ is hard to compute.
One can show, assuming HA that RSA is hard to crack. But this proof will depend on a model of security. See caveats about this on similar DH slides (bribery, timing attacks, Maginot Line).
What Could be True?

The following are all possible:

1) Factoring easy. RSA is crackable.
2) Factoring hard, HA false. RSA crackable, Factoring hard!!
3) Factoring hard, HA true, but RSA is crackable by other means. Timing Attacks. Must rethink our model of security.
4) Factoring hard, HA true, and RSA remains uncracked for years. Increases our confidence but...

Item 4 is current state with some caveats: Do Alice and Bob use it properly? Do they have large enough parameters? What is Eve's computing power?
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Plain RSA Bytes!

The RSA given above is referred to as Plain RSA. Insecure!
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**Scenario:**
Eve sees Bob send Alice $c_1$ (message is $m_1$).
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**Scenario:**
Eve sees Bob send Alice $c_1$ (message is $m_1$).
Later Eve sees Bob send Alice $c_2$ (message is $m_2$).
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What can Eve **easily** deduce?

Eve can know if $c_1 = c_2$ or not. So what?
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That alone makes it insecure.
Plain RSA is never used and should never be used!
PKCS-1.5 RSA

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We need to change how Bob sends a message;
BAD: To send \( m \in \{1, \ldots, N - 1\} \), send \( m^e \pmod{N} \).

GOOD?: To send \( m \in \{1, \ldots, N - 1\} \), pick rand \( r \), send \( (rm)^e \).
(NOTE- \( rm \) means \( r \) CONCAT with \( m \) here and elsewhere.)
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We need to change how Bob sends a message;
BAD: To send \( m \in \{1, \ldots, N - 1\} \), send \( m^e \mod N \).

GOOD?: To send \( m \in \{1, \ldots, N - 1\} \), pick rand \( r \), send \( (rm)^e \).
(NOTE- \( rm \) means \( r \) CONCAT with \( m \) here and elsewhere.)

DEC: Alice can find \( rm \) but doesn’t know divider. How to fix?
Alice and Bob agree on dividers ahead of time. Agree on
\( L_1 = \left\lfloor \frac{\log N}{3} \right\rfloor \), \( L_2 = \lfloor \log N \rfloor - L_1 \).
To send \( m \in \{0, 1\}^{L_2} \) pick random \( r \in \{0, 1\}^{L_1} \).
When Alice gets \( rm \) she will know that \( m \) is the last \( L_2 \) bits.
Example

\[ p = 31, \text{Prime } q = 37, \text{Prime } N = pq = 31 \times 37 = 1147. \]

\[ R = \phi(N) = 30 \times 36 = 1080 \]

\[ e = 77 \text{ (e rel prime to } R\text{)}, \quad d = 533 \text{ (ed } \equiv 1 \text{ (mod } R\text{))} \]

\[ L_1 = \left\lfloor \frac{\log N}{3} \right\rfloor = 3, \quad L_2 = \lfloor \log N \rfloor - L = 7. \]

Bob wants to send 1100100 (note- \( L_2 = 7 \) bits).

1. Bob generates \( L_1 = 3 \) random bits. 100.
2. Bob sends 1001100100 which is 612 in base 10 by sending 612^{77} \text{ (mod 1147)} which is 277.
3. Alice decodes by doing 277^{533} \text{ (mod 1147)} = 612
4. Alice puts 612 into binary to get 1001100100. She knows to only read the last 7 bits 1100100.

**Important:** If later Bob wants to send 100 again he will choose a DIFFERENT random 3 bits so Eve won’t know he sent the same message.
Is PKCS-1.5 RSA Secure?

VOTE

- YES (under hardness assumptions and large $n$)
- NO (there is yet another weird security thing we overlooked)

Scenario:

$N$ and $e$ are public. Bob sends $(r^m)^e \pmod{N}$. Eve cannot determine what $m$ is.

What can Eve do that is still obnoxious?

Eve can compute $2^e (r^m)^e \equiv (2^e r^m)^e \pmod{N}$. So what?

Eve can later pretend she is Bob and send $(2^e r^m)^e \pmod{N}$.

Why bad? Discuss (1) will confuse Alice (2) Sealed Bid Scenario.
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(1) will confuse Alice (2) Sealed Bid Scenario.
An encryption system is **malleable** if when Eve sees a message she can figure out a way to send a similar one, where she knows the similarity (she still does not know the message).

1. The definition above is informal.
2. Can modify RSA so that it’s probably not malleable.
3. That way is called PKCS-2.0-RSA.
4. Name BLAH-1.5 is hint that it’s not final version.
Final Points About RSA

1. PKCS-2.0-RSA is REALLY used!
2. There are many variants of RSA but all use the ideas above.
3. Factoring easy implies RSA crackable. TRUE.
4. RSA crackable implies Factoring easy: UNKNOWN.
5. RSA crackable implies Factoring easy: Often stated in expositions of crypto. They are wrong!
6. Timing attacks on RSA bypass the math.