Some Solutions to HW01 Problems

BILL, RECORD LECTURE!!!!

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Two ways to solve.
1) Write a program that goes through all $x \in \{0, \ldots, 99\}$.
2) By hand and cleverness on next slide.

Problem 2: The Clever Solutions, Mod 5

$$x^2 + 17x + 16 = (x + 16)(x + 1)$$

Lemma $(x + 1)(x + 16) \equiv 0 \pmod{100} \implies x + 1 \equiv 0 \pmod{5}$.

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$$x^2 + 17x + 16 = (x+16)(x+1)$$

Lemma $(x+1)(x+16) \equiv 0 \pmod{100} \implies x+1 \equiv 0 \pmod 5$.

Proof $x+1 \not\equiv 0 \pmod 5 \implies x+16 \not\equiv 0 \pmod 5 \implies (x+1)(x+16) \not\equiv 0 \pmod 5 \implies (x+1)(x+16) \not\equiv 0 \pmod{100}$.

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Upshot Only need to look x such that $x + 1 \equiv 0 \pmod{5}$.
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Lemma $(x+1)(x+16) \equiv 0 \pmod{100} \implies x+1 \not\equiv 3 \pmod 4$.

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Lemma $(x+1)(x+16) \equiv 0 \pmod{100} \implies x+1 \not\equiv 3 \pmod 4$.

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Upshot Only need to look at x such that $x+1 \equiv 0, 1 \pmod 4$.
Upshot Only need to look at $x \equiv 0, 3 \pmod 4$.

Problem 2. Clever Sol Cont.

1) $x \equiv 4 \pmod 5$ and $x \equiv 0 \pmod 4$ implies $x \equiv 4 \pmod{20}$.

x	$(x+1)(x+16)$	$\equiv 0 \pmod{100}$?
4	100	Y
24	1000	Y
44	2700	Y
64	5200	Y
84	8400	Y

2) $x \equiv 4 \pmod 5$ and $x \equiv 3 \pmod 4$ implies $x \equiv 19 \pmod{20}$.

x	$(x+1)(x+16)$	$\equiv 0 \pmod{100}$?
19	700	Y
39	2200	Y
59	4500	Y
79	7600	Y
99	8400	Y

SO there are 10 solutions.

Problem 2: The Point

Point of the Problem Mod 100 is very different than $\mathbb{N}$ or $\mathbb{Z}$ or even Mod 7 since you can have dth degree poly with MORE THAN d roots.

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Theorem If the domain is $\mathbb{Z}$ or $\mathbb{R}$ or $\mathbb{C}$ (the complex numbers) then every poly of degree d has $\leq d$ roots.

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Theorem If the domain is $\mathbb{Z}$ or $\mathbb{R}$ or $\mathbb{C}$ (the complex numbers) then every poly of degree d has $\le d$ roots.

The proof of this theorem used that in these domains

$$ab = 0 \implies (a = 0) \vee (b = 0)$$

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The number of b's is ALL of them: 30.

Hence there are $8 \times 30 = 240$ cool pairs.

Problem 4b

A student picks an $a, b \in \{0 \ldots, 29\}$ at random. What is the probability that (a, b) is cool relative to 30?

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$$\frac{240}{30 \times 30} = \frac{8 \times 30}{30 \times 30} = \frac{8}{30} = \frac{4}{15} \sim 0.2667$$

Problem 4c

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Hence there are $30 \times 31 = 930$ cool pairs.

Problem 4d

A student picks an $a, b \in \{0 \ldots, 30\}$ at random. What is the probability that (a, b) is cool rel to 31?
Give the answer to four decimal places.

$$\frac{930}{31 \times 31} = \frac{30 \times 31}{31 \times 31} = \frac{30}{31} =\sim 0.9677$$

Problem 4e

What types of numbers n are such that the prob of picking an (a, b) that is cool rel to n is close to 1? Give an example of a number between 1000 and 1200 where the prob is close to 1. What is the prob? Give it to 4 places.

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We want n to be PRIME. WE take $n = 1001$ which is prime. The prob of picking a cool pair is

$$\frac{1000 \times 1001}{10001 \times 1001} = \frac{1000}{1001} = 0.999.$$

Problem 4f

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A number with LOTS of prime factors. We give two examples but leave it to you to work out the answer
$n = 1024 = 2^{10}$.
$n = 4 \times 3 \times 5 \times 17$

Problem 5a

List all a, b so that the encode-key and the decode-key for affine are the same. All math is mod 26.
Need $(\forall x)[a(ax+b)+b \equiv x]$, so
$(\forall x)[a^2x+(ab+b) \equiv 1x+0]$. We match coefficients

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Case 2 $a \equiv 25$, so the $ab+b \equiv 0$ is now $25b+b \equiv 0$, so $26b \equiv 0$ OH, thats ALWAYS TRUE! So ANY b works. Pairs: $(25,b)$ for ANY $0 \le b \le 25$.

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Pairs: $(1, 0)$ $(1, 13)$, $(25, 0)$, $(25, 1)$, $\ldots$, $(25, 25)$. Note that there are 28 such pairs.

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If Eve knows Alice and Bob are doing this, the key space goes from 312 to 28. So much easier for Eve to crack the code.