BILL START RECORDING

An Early Idea on Factoring: Jevons' Number

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> **Jevons observed that there are many cases where an operation is easy but it's inverse is hard. He mentioned encryption and decryption. He mentioned multiplication and factoring. He anticipated RSA!**

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> **Can the reader say what two numbers multiplied together will produce**
>
> $$\mathbf{8,616,460,799}$$
>
> **I think it is unlikely that anyone aside from myself will ever know.**

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$$J = 8,616,460,799$$

We can now factor J easily. Was Jevons' comment stupid?

Discuss

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   Bill: How indeed!

Golomb's Method to Factor Jevons' Number

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We apply a method of Fermat (in the 1600's) to the problem of factoring J.

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To factor J find x, y such that

$$J = x^2 - y^2 = (x - y)(x + y)$$

So we must narrow our search for x, y.

Use Mods. Which Mod?

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J ends in 99. Hence

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Ah-ha. -1 is small! Mod 100 might be useful.

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Golomb's Works Mod 100

$$x^2 + 1 \equiv y^2 \pmod{100}$$

All squares mod 100:

$$\{00, 01, 04, 09, 16, 21, 24, 25, 29, 36, 41, 44, 49\} \cup$$

$$\{56, 61, 64, 69, 76, 81, 84, 89, 96\}$$

The only pairs which differ by 1 are
$(00, 01)$ and $(24, 25)$. So either:

1. $x^2 \equiv 0$, so $x \bmod 100 \in \{10, 20, 30, 40, 50, 60, 70, 80, 90\}$, OR
2. $x^2 \equiv 24$, so $x \bmod 100 \in \{18, 32, 68, 82\}$.

So

$$x \bmod 100 \in \{10, 18, 20, 30, 32, 40, 50, 60, 68, 70, 80, 90\}$$

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Since $J = x^2 - y^2$, $x^2 = J + y^2$, so

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Since $J = x^2 - y^2$, $x^2 - J = y^2$, hence

$$x^2 - J = y^2 \text{ a square}$$

Recap

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3) $x^2 - J$ is a square.

Golomb Factors Jevons' Number: $x^2 \geq J$

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92832	$1148.6\ldots$
92840	$1674.7\ldots$
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92860	$2553.1\ldots$
92868	$2829.2\ldots$
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AH-HA! We take $x = 92880$, $y = 3199$.

$$92880^2 - 3199^2 = 8,616,460,799$$

$$(92880 - 3199)(92880 + 3199) = 8,616,460,799$$

$$(89681)(96079) = 8,616,460,799$$

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A lesson for us all!

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1. Reasonable that he didn't realize that computers would get so much better.
2. Foolish since $J = 8,616,460,799$ isn't THAT big. Someone with enough determination could divide J by $2, 3, \ldots, \left\lceil \sqrt{J} \right\rceil$. This is only $\left\lceil \sqrt{J} \right\rceil = 92825$ trial divisions. Leave it to you to see if this is reasonable to finish in (say) 1 year.

Eric's Opinion of Jevons

Eric is double majoring in Math and Economics.

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When he proofread these slides he emailed me:

> *I've heard of Jevons before because he's also an economist. I am not surprised that he claimed J could not be factored, because the Modus Operandi of 19th century economists is to make bold predictions that are totally wrong.*

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- **Conclusion**
  - His arrogance: assumed the world would not change much.
  - Our arrogance: knowing how much the world did change.

Factoring Algorithms

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- We leave out the *expected time* but always mean it. Our algorithms are randomized.

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- Number Field Sieve (best known): $N^{1/L^{2/3}} = 2^{L^{1/3}}$.

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