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Reminder
Types of Attacks

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For all of these attacks. Eve's goal is to find out something about the plaintext she did not already know.

Finding out what was sent is not the only measure of success.

Learning With Errors: Private Key

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$$\equiv -840 + 137 + 149 \equiv -76 + 137 + 149 \equiv 19$$

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(Spoiler Alert: No)

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4. I know $40k_1 + 28k_2 + 111zk_3 + 7k_4 \equiv 19 \pmod{191}$ has answer $(170, 39, 3, 1)$.

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Generally:

$$(k_1, \ldots, k_n) \cdot (r_1, \ldots, r_n) = k_1 \times r_1 + \cdots + k_n \times r_n.$$

We will always be doing this Mod p.

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- Would use a bigger mod and a longer equation in real life.
- This cipher only allows transmitting one bit.

Example of Using This Cipher

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Number of possibilities for key (k_1, k_2, k_3, k_4) is now 191^3. If sees more messages can cut down search space to one possibility.

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That is **too sharp**. Instead we will do distinction between:

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We are doing it in a way that is **not used** but **better for education**.

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   $e \in^r \{-1, 0, 1\}$.

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- $e \in \{-1, 0, 1\}$. Note that $-1 \equiv 190$.

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- $e \in \{-1, 0, 1\}$. In real system $e \in \{-\gamma, \ldots, \gamma\}$, γ a param.

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- $e \in \{-1, 0, 1\}$. Note that $-1 \equiv 190$.
- $e \in \{-1, 0, 1\}$. In real system $e \in \{-\gamma, \ldots, \gamma\}$, γ a param.
- We picked 50 as our big number. In real system use $\sim \frac{p}{4}$.

Floor Ceiling Convention; Vector Notation

When we write something like $\frac{p}{4}$ where p is odd we really mean

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We will now use $\vec{r}$ for a random vector of length n.

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Is this a good cipher? Easy to use? Secure? Discuss.

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The proof that its secure uses that p is prime.

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(We will go into **why** LWE is thought to be hard when we do LWE-public, which won't be for a while.)

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   For private-key crypto, better schemes are known.

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1. A problem that plagues complexity theory is that a problem can have a bad worst-case but a reasonable average-case.
2. For LWE this is NOT an issue.
3. Hence the assumption that LWE is hard for worst case already gives you hard for avg case.

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