

BILL, RECORD LECTURE!!!!

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The Same N Attack on RSA

RSA

Let L be a security parameter

1. Alice picks two primes p, q of length L and computes $N = pq$.
2. Alice computes $\phi(N) = \phi(pq) = (p-1)(q-1)$. Denote by R .
3. Alice picks an $e \in \{\frac{R}{3}, \dots, \frac{2R}{3}\}$ that is relatively prime to R .
Alice finds d such that $ed \equiv 1 \pmod{R}$.
4. Alice broadcasts (N, e) . (Bob and Eve both see it.)
5. Bob: To send $m \in \{1, \dots, N-1\}$, send $m^e \pmod{N}$.
6. If Alice gets $m^e \pmod{N}$ she computes

$$(m^e)^d \equiv m^{ed} \equiv m^{ed \bmod R} \equiv m^{1 \bmod R} \equiv m \pmod{N}$$

Review of RSA Attacks

1. If same e , $e \leq L$. Low- e attack. **Response** Large e .
2. If same e , $m^e < N_1 \cdots N_L$. Low- e attack. **Response** Pad m .
3. NY,NY problem. Leaks info. **Response** Rand Pad m
4. Timing Attacks. **Response** Rand Pad time.

Note items 1 and 2:

e same but N 's Different

How about

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Same N , Rel Prime e 's, 2 People. Example

1. Zelda is sending messages to Alice using $(1147, 341)$
2. Zelda is sending messages to Bob using $(1147, 408)$
3. Note that 341 and 408 are relatively prime. Bad idea?

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Zelda sends m to both Alice and Bob. Eve sees

1. $m^{341} \pmod{1147}$
2. $m^{408} \pmod{1147}$

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$$1 = 56 \times 408 - 67 \times 341$$

Example Continued

1. Zelda & Alice use: (1147, 341). Zelda & Bob use (1147, 408).
2. Zelda sends m to Alice via $m^{341} \pmod{1147}$.
3. Zelda sends m to Bob via $m^{408} \pmod{1147}$.

Example Continued

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Eve does the following:

- ▶ Finds 1 as a combo of 341 and 408: $1 = 56 \times 408 - 67 \times 341$
- ▶ Find inverse of $m^{341} \pmod{1147}$. We call this m^{-341} .

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Example $27 \times 35 - 17 \times 100 + 6 \times 126 = 1$

Example Continued

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- ▶ Find inverse of $m^{100} \pmod{1147}$. We call this m^{-100} .

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Recap of What We've Done So Far

We did

1. Concrete example with Zelda sending to 2 people.
2. Concrete example with Zelda sending to 3 people.
3. General case with Zelda sending to 2 people.

We did not do

1. General case with Zelda Sending to 3 people.
2. General case with Zelda Sending to L people.

Work on the L -case is with your neighbor.

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Problem Given e_1, \dots, e_L rel prime, find $x_1, \dots, x_L \in \mathbb{Z}$ such that $\sum_{i=1}^L x_i e_i = 1$.

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What you should be thinking Bill, do an example!

An Example

Recall If a, b rel prime then exists x_1, x_2 , $ax_1 + bx_2 = 1$.

Generalization ONE Let $d = \text{GCD}(a, b)$.

Then exists x_1, x_2 such that $ax_1 + bx_2 = d$.

Good News Euclidean Alg finds d, x_1, x_2 .

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Example We find a combination of 35, 100, 126 that sums to 1.

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1. Find x_1, x_2 such that $35x_1 + 100x_2 = 5$ ($5 = \text{GCD}(35, 100)$)

$$35 \times 3 - 100 = 5$$

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1. Find x_1, x_2 such that $35x_1 + 100x_2 = 5$ ($5 = \text{GCD}(35, 100)$)

$$35 \times 3 - 100 = 5$$

2. Find y_1, y_2 such that $5y_1 + 126y_2 = 1$

$$-25 \times 5 + 126 = 1$$

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$$35 \times 3 - 100 = 5$$

2. Find y_1, y_2 such that $5y_1 + 126y_2 = 1$

$$-25 \times 5 + 126 = 1$$

- 3.

$$-25 \times (35 \times 3 - 100) + 126 = 1$$

$$-75 \times 35 + 25 \times 100 + 1 \times 126 = 1$$

Note This is diff sol than got earlier. There are many solutions.

Algorithm for x_1, x_2, x_3

This will be on a HW

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3. Randomly pad m for NY,NY problem.

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4. Randomly pad time to ward off timing attacks.

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