Homework 1

Morally Due Tue Feb 1 at 3:30PM

COURSE WEBSITE:

`http://www.cs.umd.edu/~gasarch/COURSES/752/S22/index.html`

(The symbol before gasarch is a tilde.)

1. (0 points) What is your name? Write it clearly. When is the take-home midterm due? **Learn LaTeX if you don't already know it**

2. (20 points)

   (a) (9 points) Prove that for every c, for every c coloring of $\binom{\mathsf{N}}{2}$, there is a infinite homogenous set USING a proof similar to what I did in class.

   (b) (9 points) Prove that for every c, for every c coloring of $\binom{\mathsf{N}}{2}$, there is an infinite homogenous set USING induction on c.

   (c) (2 points) Which proof do you like better? Which one do you think gives better bound when you finitize it?

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3. (20 points) Prove the following theorem rigorously (this is the infinite c-color a-ary Ramsey Theorem):

   Theorem For all $a \geq 1$, for all $c \geq 1$, and for all c-colorings of $\binom{\mathbb{N}}{a}$, there exists an infinite set $A \subseteq \mathbb{N}$ such that $\binom{A}{a}$ is monochromatic (A is an infinite homogeneous set).

   End of Statement of Theorem

   The proof should be by induction on a with the base cases being $a = 1$. You need to prove the base case.

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4. (20 points) Lets apply Ramsey Theory!

   (a) (20 points) Let

$$x_1, x_2, x_3, \ldots,$$

   be an infinite sequence of distinct reals.
   Consider the following coloring of $\binom{N}{2}$. Let $i < j$.

$$COL(i,j) = \begin{cases} RED & \text{if } x_i < x_j \\ BLUE & \text{if } x_i > x_j \end{cases} \quad (1)$$

   If you apply Ramsey Theory to this coloring you get a theorem.
   State that theorem cleanly.

   (b) (0 points, but REALLY try to do it) Prove the theorem you stated in Part a WITHOUT USING Ramsey Theory.

   (c) (0 points, but REALLY do it) Which proof do you prefer, the one that use Ramsey Theory or the one that didn't?

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5. (20 points) Lets apply Ramsey Theory!

(a) (9 points) Let

$$x_1, x_2, x_3, \ldots,$$

be an infinite sequence of points in R^2. (NOTE- these are points in R^2, not reals. So this is a different setting from the prior problem.) Consider the following coloring of $\binom{N}{2}$.

$$COL(i,j) = \begin{cases} RED & \text{if } d(x_i, x_j) > 1 \\ BLUE & \text{if } d(x_i, x_j) < 1 \\ GREEN & \text{if } d(x_i, x_j) = 1 \end{cases} \tag{2}$$

If you apply Ramsey Theory to this coloring you get a theorem. State that theorem cleanly.

(b) (9 points) Prove the theorem you stated in Part a WITHOUT USING Ramsey Theory.

(c) (2 points) Which proof do you prefer?

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6. (Extra Credit- NOT towards your grade but towards a letter I may one day write for you)

   Definition A *bipartite* graph is a graph with vertices $A \cup B$ and the only edges are between vertices of A and vertices of B. A an B can be the same set. We denote a biparatite graph with a 3-tuple (A, B, E).

   Notation $K_{n,m}$ is the bipartite graph $([n], [m], [n] \times [m])$.

   Notation $K_{\mathsf{N},\mathsf{N}}$ is the bipartite graph $(\mathsf{N}, \mathsf{N}, \mathsf{N} \times \mathsf{N})$.

   Definition If COL is a c-coloring of the edges of $K_{\mathsf{N},\mathsf{N}}$ then (H_1, H_2) is a homog set if c restricted to $H_1 \times H_2$ is constant.

   And now FINALLY the problem.

   Prove or disprove:

   For every 2-coloring of the edges of $K_{\mathsf{N},\mathsf{N}}$ there exists H_1, H_2 infinite such that (H_1, H_2) is a homog set.

7. (Extra Credit- NOT towards your grade but towards a letter I may one day write for you) Recall that the infinite Ramsey Theorem for 2-coloring the edges of a graph:

   For all colorings $COL : \binom{\mathsf{N}}{2} \rightarrow [2]$ *there exists an infinite homogenous set* $H \subseteq \mathsf{N}$.

   What if we color Z instead of N? If all we want is an *infinite homogenous set* then the exact same proof works—or you could just restrict the coloring to $\binom{\mathsf{N}}{2}$. But what if we want an infinite $H \subseteq \mathsf{Z}$ that has *the same order type as* Z?

   Definition If $(L_1, <_1)$ and $(L_2, <_2)$ are ordered sets then they are *order-equivalent* if there is a bijection f from L_1 to L_2 that preserves order. That is, $x <_1 y$ iff $f(x) <_2 f(y)$.

   And now FINALLY the problem:

   Prove or disprove:

   For all colorings $COL : \binom{\mathsf{Z}}{2} \rightarrow [2]$ *there exists a set* $H \subseteq \mathsf{Z}$ *that is order-equiv to* Z *and is homogenous.*