Finding Small Dominating Set
Via the Prob Method

William Gasarch-U of MD

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We sketch a proof that every graph with max degree $d \geq 1$ has a dominating set of size $\leq \alpha(d)n$ where $\alpha(d) < 1$.

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Hence prob $v \in Y$ is $\leq (1 - p)^{d+1}$. Hence $E(|Y|) \leq n(1 - p)^{d+1}$. Note that (1) $X \cup Y$ is a dominating set, and (2)

$$E(|X \cup Y|) = E(|X|) + E(|Y|) \leq np + n(1 - p)^{d+1}.$$

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We need to minimize the following function as $0 \leq p \leq 1$.

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$$E(|X \cup Y|) \leq n\left(\frac{\ln(d+1)+1}{d+1}\right)$$

So $\alpha(d) = \frac{\ln(d+1)+1}{d+1}$.

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How good is this? Next Slide.

Table of d:10-100

d	$\frac{\ln(d+1)+1}{d+1}$
10	0.3089
20	0.192596
30	0.143032
40	0.114965
50	0.0967025
60	0.0837848
70	0.0741223
80	0.0665981
90	0.0605589
100	0.0555953

Table of d100-1000

d	$\frac{\ln(d+1)+1}{d+1}$
100	0.0555953
200	0.0313597
300	0.0222828
400	0.0174413
500	0.0144044
600	0.0123105
700	0.0107739
800	0.00959533
900	0.00866094
1000	0.00790085

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Table of d1000-10000

d	$\frac{ln(d+1)+1}{d+1}$
1000	0.00790085
2000	0.00429855
3000	0.00300123
4000	0.00232299
5000	0.0019031
6000	0.00161634
7000	0.00140749
8000	0.00124826
9000	0.00112266
10000	0.00102094

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3. If a graph has min degree ≥ 10000 then there is DS size $\leq 0.002n$, $\frac{n}{500}$.

The Theorem Restated Completely

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Pf

Since the Expected Value of the experiment produced a set of this size, there must be some set of $\geq$ this size.

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OPT is the min size of a DS.
Alg means Poly Time Algorithm. We assume $\mathrm{P} \neq \mathrm{NP}$.

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