1. (0 points) What is your name? Write it clearly. When is the take-home midterm due? Learn LaTeX if you don’t already know it.

2. (20 points)
   
   (a) (9 points) Prove that for every \( c \), for every \( c \) coloring of \( \binom{N}{2} \), there is a homogenous set USING a proof similar to what I did in class.
   
   (b) (9 points) Prove that for every \( c \), for every \( c \) coloring of \( \binom{N}{2} \), there is an infinite homogenous set USING induction on \( c \).
   
   (c) (2 points) Which proof do you like better? Which one do you think gives better bound when you finitize it?
SOLUTION

(a) We omit since similar to what we did in class.

(b) For \( c = 1 \) this is trivial. We will also need the \( c = 2 \) cases, which we did in class.
Assume \( c \geq 3 \) and that theorem is true for \( c - 1 \). Given a \( c \)-coloring

\[
   COL : \binom{N}{2} \rightarrow [c]
\]

define

\( COL'(x, y) \) to be

i. \( COL(x, y) \) if \( COL(x, y) \in [c - 2] \).

ii. \( c - 1 \) if \( COL(x, y) \in \{c - 1, c\} \)

Note that

\[
   COL' : \binom{N}{2} \rightarrow [c - 1].
\]

Apply the Induction hypothesis to it. There are two cases.

i. There exists an infinite homog set of color one of \( \{1, \ldots, c-2\} \). Then you are done!

ii. There exists an infinite homog set of color \( c - 1 \). Let this set be \( A \). Note that if \( x, y \in A \) and \( COL'(x, y) = c - 1 \) then \( COL(x, y) \in \{c - 1, c\} \). So we do not have a homogenous set yet. But now define

\[
   COL'' : \binom{A}{2} \rightarrow \{c - 1, c\}
\]

by \( COL''(x, y) = COL(x, y) \) (we know that these are the only colors that pairs from \( A \) can have. Apply the IH with \( c = 2 \) to get a homog set.

NOTE: We went from \( c \) to \( c - 1 \). We could have gone from \( c \) to two cases of \( c/2 \) or other combinations.

(c) I prefer the proof based on what I did in class. I think it leads to better bounds in the finite cases but we’ll look at that later.
3. (20 points) Prove the following theorem rigorously (this is the infinite c-color a-ary Ramsey Theorem):

**Theorem** For all $a \geq 1$, for all $c \geq 1$, and for all $c$-colorings of $\binom{\mathbb{N}}{a}$, there exists an infinite set $A \subseteq \mathbb{N}$ such that $\binom{A}{a}$ is monochromatic ($A$ is an infinite homogeneous set).

**End of Statement of Theorem**

The proof should be by induction on $a$ with the base cases being $a = 1$. You need to prove the base case.

**SOLUTION**

Omitted- very similar to what we did in class.
4. (20 points) Let’s apply Ramsey Theory!

(a) (20 points) Let

\[ x_1, x_2, x_3, \ldots, \]

be an infinite sequence of distinct reals.
Consider the following coloring of \( \binom{\mathbb{N}}{2} \). Let \( i < j \).

\[
COL(i, j) = \begin{cases} 
RED & \text{if } x_i < x_j \\
BLUE & \text{if } x_i > x_j
\end{cases}
\]  

(1)

If you apply Ramsey Theory to this coloring you get a theorem. State that theorem cleanly.

(b) (0 points, but REALLY try to do it) Prove the theorem you stated in Part a WITHOUT USING Ramsey Theory.

(c) (0 points, but REALLY do it) Which proof do you prefer, the one that use Ramsey Theory or the one that didn’t?

**SOLUTION** Omitted— Will discuss in class.
5. (20 points) Let's apply Ramsey Theory!

(a) (9 points) Let

\[ x_1, x_2, x_3, \ldots, \]

be an infinite sequence of points in \( \mathbb{R}^2 \). (NOTE: these are points in \( \mathbb{R}^2 \), not reals. So this is a different setting from the prior problem.) Consider the following coloring of \( (\mathbb{N}^2) \).

\[
\text{COL}(i, j) = \begin{cases} 
\text{RED} & \text{if } d(x_i, x_j) > 1 \\
\text{BLUE} & \text{if } d(x_i, x_j) < 1 
\end{cases}
\]  

If you apply Ramsey Theory to this coloring you get a theorem. State that theorem cleanly.

(b) (9 points) Prove the theorem you stated in Part a WITHOUT USING Ramsey Theory.

(c) (2 points) Which proof do you prefer?

SOLUTION Omitted—Will discuss in class.
6. (Extra Credit- NOT towards your grade but towards a letter I may one day write for you)

**Definition** A bipartite graph is a graph with vertices $A \cup B$ and the only edges are between vertices of $A$ and vertices of $B$. $A$ an $B$ can be the same set. We denote a bipartite graph with a 3-tuple $(A, B, E)$.

**Notation** $K_{n,m}$ is the bipartite graph $([n], [m], [n] \times [m])$.

**Notation** $K_{N,N}$ is the bipartite graph $(N, N, N \times N)$.

**Definition** If $COL$ is a $c$-coloring of the edges of $K_{N,N}$ then $(H_1, H_2)$ is a homog set if $c$ restricted to $H_1 \times H_2$ is constant.

And now FINALLY the problem.

Prove or disprove:

*For every 2-coloring of the edges of $K_{N,N}$ there exists $H_1$, $H_2$ infinite such that $(H_1, H_2)$ is a homog set.*
7. (Extra Credit- NOT towards your grade but towards a letter I may one day write for you) Recall that the infinite Ramsey Theorem for 2-coloring the edges of a graph:

For all colorings \( \text{COL} : \left( \binom{\mathbb{N}}{2} \right) \to [2] \) there exists an infinite homogenous set \( H \subseteq \mathbb{N} \).

What if we color \( \mathbb{Z} \) instead of \( \mathbb{N} \)? If all we want is an infinite homogenous set then the exact same proof works—or you could just restrict the coloring to \( \binom{\mathbb{N}}{2} \). But what if we want an infinite \( H \subseteq \mathbb{Z} \) that has the same order type as \( \mathbb{Z} \)?

**Definition** If \( (L_1, <_1) \) and \( (L_2, <_2) \) are ordered sets then they are order-equivalent if there is a bijection \( f \) from \( L_1 \) to \( L_2 \) that preserves order. That is, \( x <_1 y \) iff \( f(x) <_2 f(y) \).

And now FINALLY the problem:

Prove or disprove:

For all colorings \( \text{COL} : \left( \binom{\mathbb{Z}}{2} \right) \to [2] \) there exists a set \( H \subseteq \mathbb{Z} \) that is order-equiv to \( \mathbb{Z} \) and is homogenous.