**Homework 1**

Morally Due Tue Feb 8 at 3:30PM. Dead Cat Feb 10 at 3:30

COURSE WEBSITE:

`http://www.cs.umd.edu/~gasarch/COURSES/752/S22/index.html`

(The symbol before gasarch is a tilde.)

1. (0 points) What is your name? Write it clearly. When is the take-home midterm due? **Learn LaTeX if you don't already know it**

2. (20 points)

   (a) (9 points) Prove that for every $c$, for every $c$ coloring of $\binom{\mathsf{N}}{2}$, there is a homogenous set USING a proof similar to what I did in class.

   (b) (9 points) Prove that for every $c$, for every $c$ coloring of $\binom{\mathsf{N}}{2}$, there is an infinite homogenous set USING induction on $c$.

   (c) (2 points) Which proof do you like better? Which one do you think gives better bound when you finitize it?

**SOLUTION**

(a) We omit since similar to what we did in class.

(b) For $c = 1$ this is trivial. We will also need the $c = 2$ cases, which we did in class.

Assume $c \geq 3$ and that theorem is true for $c-1$. Given a $c$-coloring

$$COL : \binom{\mathsf{N}}{2} \rightarrow [c]$$

define

$COL'(x, y)$ to be

i. $COL(x, y)$ if $COL(x, y) \in [c - 2]$.

ii. $c - 1$ if $COL(x, y) \in \{c - 1, c\}$

Note that

$$COL' : \binom{\mathsf{N}}{2} \rightarrow [c - 1].$$

Apply the Induction hypothesis to it. There are two cases.

i. There exists an infinite homog set of color one of $\{1, \ldots, c-2\}$. Then you are done!

ii. There exists an infinite homog set of color $c - 1$. Let this set be $A$. Note that if $x, y \in A$ and $COL'(x, y) = c - 1$ then $COL(x, y) \in \{c - 1, c\}$. So we do not have a homogenous set yet. But now define

$$COL'' : \binom{A}{2} \rightarrow \{c - 1, c\}$$

by $COL''(x, y) = COL(x, y)$ (we know that these are the only colors that pairs from $A$ can have. Apply the IH with $c = 2$ to get a homog set.

NOTE: We went from $c$ to $c - 1$. We could have gone from $c$ to two cases of $c/2$ or other combinations.

(c) I prefer the proof based on what I did in class. I think it leads to better bounds in the finite cases but we'll look at that later.

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3. (20 points) Prove the following theorem rigorously (this is the infinite $c$-color $a$-ary Ramsey Theorem):

   **Theorem** For all $a \ge 1$, for all $c \ge 1$, and for all $c$-colorings of $\binom{\mathbb{N}}{a}$, there exists an infinite set $A \subseteq \mathbb{N}$ such that $\binom{A}{a}$ is monochromatic ($A$ is an infinite homogeneous set).

   **End of Statement of Theorem**

   The proof should be by induction on $a$ with the base cases being $a = 1$. You need to prove the base case.

   **SOLUTION**

   Omitted- very similar to what we did in class.

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4. (20 points) Lets apply Ramsey Theory!

   (a) (20 points) Let

   $$x_1, x_2, x_3, \ldots,$$

   be an infinite sequence of distinct reals.
   Consider the following coloring of $\binom{N}{2}$. Let $i < j$.

   $$COL(i,j) = \begin{cases} RED & \text{if } x_i < x_j \\ BLUE & \text{if } x_i > x_j \end{cases} \tag{1}$$

   If you apply Ramsey Theory to this coloring you get a theorem. State that theorem cleanly.

   (b) (0 points, but REALLY try to do it) Prove the theorem you stated in Part a WITHOUT USING Ramsey Theory.

   (c) (0 points, but REALLY do it) Which proof do you prefer, the one that use Ramsey Theory or the one that didn't?

**SOLUTION** Omitted— Will discuss in class.

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5. (20 points) Lets apply Ramsey Theory!

   (a) (9 points) Let

$$x_1, x_2, x_3, \ldots,$$

   be an infinite sequence of points in $\mathsf{R}^2$. (NOTE- these are points in $\mathsf{R}^2$, not reals. So this is a different setting from the prior problem.) Consider the following coloring of $\binom{N}{2}$.

$$COL(i,j) = \begin{cases} RED & \text{if } d(x_i, x_j) > 1 \\ BLUE & \text{if } d(x_i, x_j) < 1 \end{cases} \tag{2}$$

   If you apply Ramsey Theory to this coloring you get a theorem. State that theorem cleanly.

   (b) (9 points) Prove the theorem you stated in Part a WITHOUT USING Ramsey Theory.

   (c) (2 points) Which proof do you prefer?

**SOLUTION** Omitted— Will discuss in class.

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6. (Extra Credit- NOT towards your grade but towards a letter I may one day write for you)

   *Definition* A *bipartite* graph is a graph with vertices $A \cup B$ and the only edges are between vertices of $A$ and vertices of $B$. $A$ an $B$ can be the same set. We denote a biparatite graph with a 3-tuple $(A, B, E)$.

   *Notation* $K_{n,m}$ is the bipartite graph $([n], [m], [n] \times [m])$.

   *Notation* $K_{\mathsf{N},\mathsf{N}}$ is the bipartite graph $(\mathsf{N}, \mathsf{N}, \mathsf{N} \times \mathsf{N})$.

   *Definition* If $COL$ is a $c$-coloring of the edges of $K_{\mathsf{N},\mathsf{N}}$ then $(H_1, H_2)$ is a homog set if $c$ restricted to $H_1 \times H_2$ is constant.

   And now FINALLY the problem.

   Prove or disprove:

   *For every 2-coloring of the edges of* $K_{\mathsf{N},\mathsf{N}}$ *there exists* $H_1$, $H_2$ *infinite such that* $(H_1, H_2)$ *is a homog set.*

7. (Extra Credit- NOT towards your grade but towards a letter I may one day write for you) Recall that the infinite Ramsey Theorem for 2-coloring the edges of a graph:

   *For all colorings* $COL : \binom{\mathsf{N}}{2} \rightarrow [2]$ *there exists an infinite homogenous set* $H \subseteq \mathsf{N}$.

   What if we color $\mathsf{Z}$ instead of $\mathsf{N}$? If all we want is an *infinite homogenous set* then the exact same proof works—or you could just restrict the coloring to $\binom{\mathsf{N}}{2}$. But what if we want an infinite $H \subseteq \mathsf{Z}$ that has *the same order type as* $\mathsf{Z}$?

   **Definition** If $(L_1, <_1)$ and $(L_2, <_2)$ are ordered sets then they are *order-equivalent* if there is a bijection $f$ from $L_1$ to $L_2$ that preserves order. That is, $x <_1 y$ iff $f(x) <_2 f(y)$.

   And now FINALLY the problem:

   Prove or disprove:

   *For all colorings* $COL : \binom{\mathsf{Z}}{2} \rightarrow [2]$ *there exists a set* $H \subseteq \mathsf{Z}$ *that is order-equiv to* $\mathsf{Z}$ *and is homogenous.*