HW 01 Some Solutions

William Gasarch-U of MD
Problem 2

1. Prove that for every $c$, for every $c$ coloring of $\binom{\mathbb{N}}{2}$, there is a homogenous set USING a proof similar to what I did in class.

SKETCH
When processing a node $x_i$ instead of saying Either an inf numb of $R$ or $B$ edges come out of $x_i$, say Either an inf numb of $R_1$ or $\cdots$ or $R_c$ edges come out of $x_i$.

2. Prove that for every $c$, for every $c$ coloring of $\binom{\mathbb{N}}{2}$, there is an inf homogenous set USING induction on $c$.

SKETCH
$c = 1$ trivial.
Assume for $c$. When $c$-color $\binom{\mathbb{N}}{2}$ with colors $\{1, \ldots, c\}$ view it as $c - 1$ colors: $1, 2, \ldots, c - 2$ and color $\{c - 1, c\}$ for those edges colored EITHER. Get homog set. If its Homog with color 1 or $\cdots$ $c - 2$ then done. If its homog color $\{c - 1, c\}$ then use 2-color case.
1. Prove that for every $c$, for every $c$ coloring of $\binom{\mathbb{N}}{2}$, there is a homogenous set USING a proof similar to what I did in class. **Sketch** When processing a node $x_i$ instead of saying *Either an inf numb of R or B edges come out of $x_i$.*, say

*Either an inf numb of $R_1$ or $\cdots$ or $R_c$ edges come out of $x_i$.***
Problem 2

1. Prove that for every $c$, for every $c$ coloring of $\binom{\mathbb{N}}{2}$, there is a homogenous set USING a proof similar to what I did in class. **SKETCH** When processing a node $x_i$ instead of saying

   *Either an inf numb of $R$ or $B$ edges come out of $x_i$.*

   say

   *Either an inf numb of $R_1$ or $\cdots$ or $R_c$ edges come out of $x_i$.*

2. Prove that for every $c$, for every $c$ coloring of $\binom{\mathbb{N}}{2}$, there is an inf homogenous set USING induction on $c$. 
Problem 2

1. Prove that for every $c$, for every $c$ coloring of $\binom{\mathbb{N}}{2}$, there is a homogenous set USING a proof similar to what I did in class. **SKETCH** When processing a node $x_i$ instead of saying

_Either an inf numb of R or B edges come out of $x_i$._

say

_Either an inf numb of $R_1$ or $\cdots$ or $R_c$ edges come out of $x_i$._

2. Prove that for every $c$, for every $c$ coloring of $\binom{\mathbb{N}}{2}$, there is an inf homogenous set USING induction on $c$. **SKETCH** $c = 1$ trivial.
Problem 2

1. Prove that for every $c$, for every $c$ coloring of $\binom{\mathbb{N}}{2}$, there is a homogenous set USING a proof similar to what I did in class. **SKETCH** When processing a node $x_i$ instead of saying
   *Either an inf numb of $R$ or $B$ edges come out of $x_i$.*
   say
   *Either an inf numb of $R_1$ or $\cdots$ or $R_c$ edges come out of $x_i$.*

2. Prove that for every $c$, for every $c$ coloring of $\binom{\mathbb{N}}{2}$, there is an inf homogenous set USING induction on $c$. **SKETCH** $c = 1$ trivial. Assume for $c$. 
Problem 2

1. Prove that for every $c$, for every $c$ coloring of $\binom{\mathbb{N}}{2}$, there is a homogenous set USING a proof similar to what I did in class. **SKETCH** When processing a node $x_i$ instead of saying *Either an inf numb of R or B edges come out of $x_i$.* say *Either an inf numb of $R_1$ or $\cdots$ or $R_c$ edges come out of $x_i$.*

2. Prove that for every $c$, for every $c$ coloring of $\binom{\mathbb{N}}{2}$, there is an inf homogenous set USING induction on $c$. **SKETCH** $c = 1$ trivial. Assume for $c$. When $c$-color $\binom{\mathbb{N}}{2}$ with colors $\{1, \ldots, c\}$ view it as $c - 1$ colors: $1, 2, \ldots, c - 2$ and color $\{c - 1, c\}$ for those edges colored EITHER. Get homog set.
Problem 2

1. Prove that for every $c$, for every $c$ coloring of $\binom{\mathbb{N}}{2}$, there is a homogenous set USING a proof similar to what I did in class. 
   **SKETCH** When processing a node $x_i$ instead of saying
   
   *Either an inf numb of $R$ or $B$ edges come out of $x_i$.*
   
   say
   
   *Either an inf numb of $R_1$ or $\cdots$ or $R_c$ edges come out of $x_i$.*

2. Prove that for every $c$, for every $c$ coloring of $\binom{\mathbb{N}}{2}$, there is an inf homogenous set USING induction on $c$.
   **SKETCH** $c = 1$ trivial. Assume for $c$.
   When $c$-color $\binom{\mathbb{N}}{2}$ with colors $\{1, \ldots, c\}$ view it as $c - 1$ colors:
   
   $1, 2, \ldots, c - 2$ and color $\{c - 1, c\}$ for those edges colored EITHER. Get homog set.
   
   If its Homog with color 1 or $\cdots$ $c - 2$ then done.
Problem 2

1. Prove that for every \( c \), for every \( c \) coloring of \( \binom{N}{2} \), there is a homogenous set USING a proof similar to what I did in class.

**SKETCH** When processing a node \( x_i \) instead of saying

*Either an inf numb of R or B edges come out of \( x_i \).*

say

*Either an inf numb of R\(_1\) or \cdots\) or R\(_c\) edges come out of \( x_i \).*

2. Prove that for every \( c \), for every \( c \) coloring of \( \binom{N}{2} \), there is an inf homogenous set USING induction on \( c \).

**SKETCH** \( c = 1 \) trivial. Assume for \( c \).

When \( c \)-color \( \binom{N}{2} \) with colors \( \{1, \ldots, c\} \) view it as \( c - 1 \) colors:

1, 2, \ldots, \( c - 2 \) and color \( \{c - 1, c\} \) for those edges colored EITHER. Get homog set.

If its Homog with color 1 or \( \cdots\) \( c - 2 \) then done.

If its homog color \( \{c - 1, c\} \) then use 2-color case.
Problem 2

1. Prove that for every $c$, for every $c$ coloring of $\binom{\mathbb{N}}{2}$, there is a homogenous set USING a proof similar to what I did in class. **SKETCH** When processing a node $x_i$ instead of saying
   *Either an inf numb of R or B edges come out of $x_i$.*
   say
   *Either an inf numb of $R_1$ or $\cdots$ or $R_c$ edges come out of $x_i$.*

2. Prove that for every $c$, for every $c$ coloring of $\binom{\mathbb{N}}{2}$, there is an inf homogenous set USING induction on $c$. **SKETCH** $c = 1$ trivial. Assume for $c$.
   When $c$-color $\binom{\mathbb{N}}{2}$ with colors $\{1, \ldots, c\}$ view it as $c - 1$ colors:
   $1, 2, \ldots, c - 2$ and color $\{c - 1, c\}$ for those edges colored EITHER. Get homog set.
   If its Homog with color 1 or $\cdots$ $c - 2$ then done.
   If its homog color $\{c - 1, c\}$ then use 2-color case.

**VOTE** Which proof did you like better.
A Subtle Point that I will not take off points for.
I didn’t realize it myself until a student asked me about it.
Problem 2- A Subtle Point

A Subtle Point that I will not take off points for. I didn’t realize it myself until a student asked me about it.

When doing the case where color \{c - 1, c\} occurs infinitely often we use 2-ary Ramsey.
A Subtle Point that I will not take off points for. I didn’t realize it myself until a student asked me about it.

When doing the case where color \( \{c - 1, c\} \) occurs inf often we use 2-ary Ramsey.

So I am using the theorem
\[
(\forall) \text{COL}: \binom{\mathbb{N}}{2} \rightarrow [2] \ (\exists) \text{inf homog set}.
\]
Problem 2- A Subtle Point

A Subtle Point that I \textbf{will not} take off points for. I didn’t realize it myself until a student asked me about it.

When doing the case where color \( \{c - 1, c\} \) occurs \textit{inf} often we use 2-ary Ramsey.

So I am using the theorem
\[
(\forall) \text{COL}: \binom{\mathbb{N}}{2} \rightarrow [2] (\exists) \text{inf homog set}.
\]

NO, I am not using that! The set I am coloring is an infinite subset of \( \mathbb{N} \). So I am really using the following trivial corollary of the above theorem:
\[
(\forall) \text{inf} A \subseteq \mathbb{N}, (\forall) \text{COL}: \binom{A}{2} \rightarrow [2] (\exists) \text{inf homog set}.
\]
Proof for $a$-ary $c$-color Ramsey.

**SKETCH** Given $\text{COL}: \binom{\mathbb{N}}{a} \to [c]$, form $\text{COL}' : \binom{\mathbb{N}}{a-1} \to [c-1]$ via

$$\text{COL}'(z_1, \ldots, z_{a-1}) = \text{COL}(x_1, z_1, \ldots, z_{a-1}).$$
Problem 3

Proof for $a$-ary $c$-color Ramsey.

**SKETCH** Given $\text{COL} : \binom{\mathbb{N}}{a} \rightarrow [c]$, form $\text{COL}' : \binom{\mathbb{N}}{a-1} \rightarrow [c-1]$ via

$$\text{COL}'(z_1, \ldots, z_{a-1}) = \text{COL}(x_1, z_1, \ldots, z_{a-1}).$$

Find homog set inductively and kill all vertices not in that set.
Proof for $a$-ary $c$-color Ramsey.

**SKETCH** Given $\text{COL}: \binom{\mathbb{N}}{a} \rightarrow [c]$, form $\text{COL}' : \binom{\mathbb{N}}{a-1} \rightarrow [c - 1]$ via

$$\text{COL}'(z_1, \ldots, z_{a-1}) = \text{COL}(x_1, z_1, \ldots, z_{a-1}).$$

Find homog set inductively and kill all vertices not in that set. $x_2$ is min element of homog set.
Problem 3

Proof for \(a\)-ary \(c\)-color Ramsey.

**SKETCH** Given \(\text{COL}: \binom{\mathbb{N}}{a} \rightarrow [c]\), form \(\text{COL}' : \binom{\mathbb{N}}{a-1} \rightarrow [c-1]\) via

\[
\text{COL}'(z_1, \ldots, z_{a-1}) = \text{COL}(x_1, z_1, \ldots, z_{a-1}).
\]

Find homog set inductively and kill all vertices not in that set.

\(x_2\) is min element of homog set.

Lather, Rinse, Repeat to get \(x_1, x_2, \ldots\).
Problem 4 (slightly modified)

$x_1, x_2, x_3, \ldots$ is an inf seq of reals.
Problem 4 (slightly modified)

$x_1, x_2, x_3, \ldots$ is an inf seq of reals.
For $i < j$.

\begin{equation}
COL(i, j) = \begin{cases} 
    RED & \text{if } x_i < x_j \\
    BLUE & \text{if } x_i > x_j \\
    GREEN & \text{if } x_i = x_j 
\end{cases}
\end{equation} (1)

Apply Ramsey Theory to get a theorem.
If homog RED then get subseq set $x_{i_1} < x_{i_2} < \ldots$
If homog BLUE then get subseq set $x_{i_1} > x_{i_2} > \ldots$
If homog GREEN then get subseq set $x_{i_1} = x_{i_2} = \ldots$

Thm
Every inf seq of $\mathbb{R}$ has either an inf $\uparrow$ seq, an inf $\downarrow$ seq, or an inf = seq.
Problem 4 (slightly modified)

\( x_1, x_2, x_3, \ldots \) is an inf seq of reals.
For \( i < j \).

\[
\text{COL}(i, j) = \begin{cases} 
    \text{RED} & \text{if } x_i < x_j \\
    \text{BLUE} & \text{if } x_i > x_j \\
    \text{GREEN} & \text{if } x_i = x_j
\end{cases}
\]  

Apply Ramsey Theory to get a theorem.
Problem 4 (slightly modified)

$x_1, x_2, x_3, \ldots$ is an inf seq of reals.
For $i < j$.

$$COL(i, j) = \begin{cases} 
RED & \text{if } x_i < x_j \\
BLUE & \text{if } x_i > x_j \\
GREEN & \text{if } x_i = x_j 
\end{cases} \quad (1)$$

Apply Ramsey Theory to get a theorem.
If homog RED then get subseq set $x_{i_1} < x_{i_2} < \ldots$
If homog BLUE then get subseq set $x_{i_1} > x_{i_2} > \ldots$
If homog GREEN then get subseq set $x_{i_1} = x_{i_2} = \ldots$
Problem 4 (slightly modified)

\( x_1, x_2, x_3, \ldots \) is an inf seq of reals.
For \( i < j \).

\[ \text{COL}(i, j) = \begin{cases} 
\text{RED} & \text{if } x_i < x_j \\
\text{BLUE} & \text{if } x_i > x_j \\
\text{GREEN} & \text{if } x_i = x_j 
\end{cases} \]  \hspace{1em} (1)

Apply Ramsey Theory to get a theorem.
If homog RED then get subseq set \( x_{i_1} < x_{i_2} < \ldots \)
If homog BLUE then get subseq set \( x_{i_1} > x_{i_2} > \ldots \)
If homog GREEN then get subseq set \( x_{i_1} = x_{i_2} = \ldots \)

**Thm** Every inf seq of \( R \) has either an inf \( \uparrow \) seq, an inf \( \downarrow \) seq, or an inf = seq.
Can generalize to \( \mathbb{R}^n \) by either applying Ramsey with 2-colors \( n \) times, or applying Ramsey with \( 3^n \) colors.

\textbf{Thm} Every inf seq of \( \mathbb{R}^n \) has an inf subseq where, for each coordinate, either ↑ seq, or ↓ or =.
Can generalize to $\mathbb{R}^n$ by either applying Ramsey with 2-colors $n$ times, or applying Ramsey with $3^n$ colors.

**Thm** Every inf seq of $\mathbb{R}^n$ has an inf subseq where, for each coordinate, either ↑ seq, or ↓ or =.

This is a part of the proof of the Bolzano-Weierstrass Theorem. Next Slide.
Lemma

1. Any increasing sequence bounded sequence of reals converges to a real.
2. Any decreasing sequence bounded sequence of reals converges to a real.

This is not obvious. This depends on the construction of the Reals.
Bolzano-Weierstrass Theorem

Lemma

1. Any increasing sequence bounded sequence of reals converges to a real.
2. Any decreasing sequence bounded sequence of reals converges to a real.

This is not obvious. This depends on the construction of the Reals.

BW Thm If \( p_1, p_2, p_3, \ldots \) is an inf sequence of points in \( \mathbb{R}^n \) that is contained in a box, then there exists a subsequence that converges to a point in \( \mathbb{R}^n \).
**Bolzano-Weierstrass Theorem**

**Lemma**

1. Any increasing sequence bounded sequence of reals converges to a real.

2. Any decreasing sequence bounded sequence of reals converges to a real.

This is not obvious. This depends on the construction of the Reals.

**BW Thm** If \( p_1, p_2, p_3, \ldots \) is an inf sequence of points in \( \mathbb{R}^n \) that is contained in a box, then there exists a subsequence that converges to a point in \( \mathbb{R}^n \).

**Proof**

Problem 4 yields that there is a subsequence in each coordinate that is either \( \downarrow, \uparrow, \) or \( = \). Lemma yields each coord converges.
The BW thm was proven in 1817, way before Ramsey’s Theorem.
Problem 5- History

The BW thm was proven in 1817, way before Ramsey’s Theorem.

The proof used a Ramsey-like argument.
Problem 5- History

The BW thm was proven in 1817, way before Ramsey’s Theorem.

The proof used a Ramsey-like argument.

Our approach is cleaner.
The BW thm was proven in 1817, way before Ramsey’s Theorem.

The proof used a Ramsey-like argument.

Our approach is cleaner.

When I first taught this application 4 years ago I Googled Bolzano-Weierstrass to get more information about this.
Problem 5- History

The BW thm was proven in 1817, way before Ramsey’s Theorem.

The proof used a Ramsey-like argument.

Our approach is cleaner.

When I first taught this application 4 years ago I Googled Bolzano-Weierstrass to get more information about this.

Google, knowing that I collect Math Novelty Songs, completed it to Bolzano-Weierstrass Rap which I then added to my collection.
The BW thm was proven in 1817, way before Ramsey’s Theorem.

The proof used a Ramsey-like argument.

Our approach is cleaner.

When I first taught this application 4 years ago I Googled Bolzano-Weierstrass to get more information about this.

Google, knowing that I collect Math Novelty Songs, completed it to Bolzano-Weierstrass Rap which I then added to my collection.

It is the worst math novelty song ever. Listen for yourself: https://www.youtube.com/watch?v=df018klwKHg
Problem 5

\[ p_1, p_2, p_3, \ldots, \]

be an infinite sequence of points in \( \mathbb{R}^2 \).
Consider the following coloring of \( \binom{\mathbb{N}}{2} \).

\[
COL(i, j) = \begin{cases} 
RED & \text{if } d(p_i, p_j) > 1 \\
BLUE & \text{if } d(p_i, p_j) < 1 
\end{cases}
\] (2)

Apply Ramsey Theorem. What do you get?

**SOLUTION**

**Thm** Given an infinite sequence of points in \( \mathbb{R}^2 \) there exists an infinite subset so that either (a) they are all within 1 of each other, or (b) they are all more than 1 apart.
Problem 4 and 5 thoughts

The proofs of the theorems in Problem 4 and 5 are FAR EASIER with Ramsey Theory. The proofs without Ramsey end up doing Ramsey in context.
Problem 6 (Extra Credit)

Prove or disprove:

For every 2-coloring of the edges of $K_{\mathbb{N},\mathbb{N}}$ there exists $H_1, H_2$ infinite such that $(H_1, H_2)$ is a homog set.
Problem 6 (Extra Credit)

Prove or disprove:
For every 2-coloring of the edges of $K_{\mathbb{N}, \mathbb{N}}$ there exists $H_1$, $H_2$ infinite such that $(H_1, H_2)$ is a homog set.

Discuss and Vote
Problem 6 (Extra Credit)

Prove or disprove:

For every 2-coloring of the edges of $K_{\mathbb{N},\mathbb{N}}$ there exists $H_1, H_2$ infinite such that $(H_1, H_2)$ is a homog set.

Discuss and Vote

SOLUTION FALSE. Color with

$$\text{COL}(i, j) = \begin{cases} 
\text{RED} & \text{if } i < j \\
\text{BLUE} & \text{if } i \geq j 
\end{cases}$$
Problem 6 (Future Extra Credit)

Thought What if we use 100 colors? The same counterexample works but you end up with an \((H_1, H_2)\) homog set that only has TWO colors. We will call that a 2-homog set.
Thought What if we use 100 colors? The same counterexample works but you end up with an \((H_1, H_2)\) homog set that only has TWO colors. We will call that a 2-homog set.

Prove or disprove:

For every 100-coloring of the edges of \(K_{\mathbb{N}, \mathbb{N}}\) there exists \(H_1, H_2\) infinite such that \((H_1, H_2)\) is a 2-homog set. 3-homog set(?)

Some \(c\)-homog with \(c < 100\)?
Problem 7 (Extra Credit)

Prove or disprove:

For all colorings $\text{COL} : (\mathbb{Z}/2) \rightarrow [2]$ there exists a set $H \subseteq \mathbb{Z}$ that is order-equiv to $\mathbb{Z}$ and is homogenous.
Problem 7 (Extra Credit)

Prove or disprove:
For all colorings \( \text{COL} : \mathbb{Z}_2 \rightarrow [2] \) there exists a set \( H \subseteq \mathbb{Z} \) that is order-equiv to \( \mathbb{Z} \) and is homogenous.

**Discuss and Vote**

\[
\begin{aligned}
\text{SOLUTION} & \quad \text{FALSE. Color with} \\
\text{COL}(i, j) & = \begin{cases} 
\text{RED} & \text{if } i, j \geq 0 \\
\text{BLUE} & \text{if } i, j < 0 \\
\text{BLUE} & \text{if one is } \geq 0 \text{ and the other is } < 0 
\end{cases}
\end{aligned}
\]
Problem 7 (Extra Credit)

Prove or disprove:
For all colorings \( \text{COL} : (\mathbb{Z}/2) \to [2] \) there exists a set \( H \subseteq \mathbb{Z} \) that is order-equiv to \( \mathbb{Z} \) and is homogenous.

Discuss and Vote
SOLUTION FALSE. Color with

\[
\text{COL}(i, j) = \begin{cases} 
  \text{RED} & \text{if } i, j \geq 0 \\
  \text{BLUE} & \text{if } i, j < 0 \\
  \text{BLUE} & \text{if one is } \geq 0 \text{ and the other is } < 0 
\end{cases}
\]
Thought What if we use 100 colors? The same counterexample works but you end up with an $H$ homog set that only has TWO colors. We will call that a 2-homog set.
Thought  What if we use 100 colors? The same counterexample works but you end up with an $H$ homog set that only has TWO colors. We will call that a 2-homog set.

Prove or disprove:

For every 100-coloring of the edges of $K_Z$ there exists 2-homog $H$ that is order-isom to $Z$. 3-homog. Some $c$-homog with $c < 100$?