HW 02 Some Solutions

William Gasarch-U of MD

Problem 2

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SOL $H = \{a_1 < a_2 < \cdots \}$.
We take $H_* = \{a_2, a_3, \cdots \}$. We show H_* is COL-homog.

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From $\mathrm{COL}'(a_1, a_{j_1}, a_{j_2}, a_k)$ we know $\mathrm{COL}(a_1, a_k) = \mathrm{COL}(a_{j_1}, a_{j_2})$.
Hence $\mathrm{COL}(a_{i_1}, a_{i_2}) = \mathrm{COL}(a_{j_1}, a_{j_2})$.

Problem 2 (misc)

Could we have left a_1 in?
We have:

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Can check it satisfies condition.
Easily seen to not be homog.

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$\deg_c(x) \leq 1$. *If* M *is MAXIMAL rainb then* $|M| \geq \Omega(f(n))$.

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Map $X - M$ into $\binom{M}{2} \times M$:

$x \in X - M$ maps to $(\{p, q\}, r)$ with $\mathrm{COL}(x, r) = \mathrm{COL}(p, q)$.

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Finish on next slide.

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$$|X - M| \leq \left| M \times \binom{M}{2} \right| \leq \frac{M^3}{2} \leq \frac{f(n)^3}{2}.$$

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Take any $A = 1$ to get a contradiction (could do slightly better).
$f(n) = n^{1/3}$.

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The number of reals in the image of the colorings is countable so we can apply Can Ramsey. When we apply it we find that there is a set $H \subseteq \mathsf{N}$, $|H| = \infty$ that is either homog, max-homog, min-homog, or rainb. We show H rainb, so all distances different.

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Three cases: homog, min-homog, max-homog.

H is homog.

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Exercise: cannot have 4 points in the plane with all distances the same.

Problem 4, H Min-Homog

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So $p_2, p_3, \ldots$ are on a circle centered at p_1.

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Combine: p_3, p_4, p_5 are all on both a circle centered at p_1 and a circle centered at p_2.

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Combine: p_3, p_4, p_5 are all on both a circle centered at p_1 and a circle centered at p_2.

But two circles with diff centers intersect in at most 2 points. Contradiction.

$(\forall i, j \leq 4)[|p_4 - p_i| = |p_4 - p_j|]$.
So p_1, p_2, p_3 are on a circle centered at p_4.

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So p_1, p_2, p_3, p_4 are on a circle centered at p_5.

So p_1, p_2, p_3 are on a circle centered at p_4 and p_5.

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End of Proof of Theorem