HW06 Some Solutions

William Gasarch-U of MD

Problem 3

$(\forall k)(\exists n)(\forall \mathrm{COL}: \binom{\{k,\ldots,n\}}{1} \to \omega$ *either*

- $\exists$ *a LARGE homog set (LHS), or*
- $\exists$ *a LARGE rainbow set (LRS)*

Problem 3

$(\forall k)(\exists n)(\forall \mathrm{COL} : \binom{\{k,\ldots,n\}}{1} \to \omega$ *either*

- $\exists$ *a LARGE homog set (LHS), or*
- $\exists$ *a LARGE rainbow set (LRS)*

n is least such that $(\forall \mathrm{COL}' : \binom{\{k,\ldots,n\}}{2} \to [2])(\exists LHS)$.

Problem 3

$(\forall k)(\exists n)(\forall \mathrm{COL} : \binom{\{k,\ldots,n\}}{1} \rightarrow \omega$ *either*

- $\exists$ *a LARGE homog set (LHS), or*
- $\exists$ *a LARGE rainbow set (LRS)*

n is least such that $(\forall \mathrm{COL}' : \binom{\{k,\ldots,n\}}{2} \rightarrow [2])(\exists LHS)$.
Let $COL' : \binom{\{k,\ldots,n\}}{2} \rightarrow [2]$ via:

Problem 3

$(\forall k)(\exists n)(\forall \mathrm{COL} : \binom{\{k,\ldots,n\}}{1} \to \omega$ *either*

- $\exists$ *a LARGE homog set (LHS), or*
- $\exists$ *a LARGE rainbow set (LRS)*

n is least such that $(\forall \mathrm{COL}' : \binom{\{k,\ldots,n\}}{2} \to [2])(\exists LHS)$.
Let $COL' : \binom{\{k,\ldots,n\}}{2} \to [2]$ via:

$$COL'(x,y) = \begin{cases} EQ & \text{if } COL(x) = COL(y) \\ NEQ & \text{if } COL(x) \neq COL(y) \end{cases}$$

Problem 3

$(\forall k)(\exists n)(\forall \mathrm{COL} : \binom{\{k,\ldots,n\}}{1} \rightarrow \omega$ *either*

- $\exists$ *a LARGE homog set (LHS), or*
- $\exists$ *a LARGE rainbow set (LRS)*

n is least such that $(\forall \mathrm{COL}' : \binom{\{k,\ldots,n\}}{2} \rightarrow [2])(\exists LHS)$.
Let $COL' : \binom{\{k,\ldots,n\}}{2} \rightarrow [2]$ via:

$$COL'(x,y) = \begin{cases} EQ & \text{if } COL(x) = COL(y) \\ NEQ & \text{if } COL(x) \neq COL(y) \end{cases}$$

Apply Large Ramsey theorem to get a Large Homog Set

$$\{h_1 < h_2 < \cdots < h_{h_1+1}\}$$

Problem 3

$(\forall k)(\exists n)(\forall \mathrm{COL} : \binom{\{k,\ldots,n\}}{1} \to \omega$ *either*

- $\exists$ *a LARGE homog set (LHS), or*
- $\exists$ *a LARGE rainbow set (LRS)*

n is least such that $(\forall \mathrm{COL}' \colon \binom{\{k,\ldots,n\}}{2} \to [2])(\exists LHS)$.
Let $COL' : \binom{\{k,\ldots,n\}}{2} \to [2]$ via:

$$COL'(x,y) = \begin{cases} EQ & \text{if } COL(x) = COL(y) \\ NEQ & \text{if } COL(x) \neq COL(y) \end{cases}$$

Apply Large Ramsey theorem to get a Large Homog Set

$$\{h_1 < h_2 < \cdots < h_{h_1+1}\}$$

If COL'-homog color is EQ then H is homog for COL.

Problem 3

$(\forall k)(\exists n)(\forall \mathrm{COL} : \binom{\{k,\ldots,n\}}{1} \rightarrow \omega$ *either*

- $\exists$ *a LARGE homog set (LHS), or*
- $\exists$ *a LARGE rainbow set (LRS)*

n is least such that $(\forall \mathrm{COL}' \colon \binom{\{k,\ldots,n\}}{2} \rightarrow [2])(\exists LHS)$.
Let $COL' : \binom{\{k,\ldots,n\}}{2} \rightarrow [2]$ via:

$$COL'(x,y) = \begin{cases} EQ & \text{if } COL(x) = COL(y) \\ NEQ & \text{if } COL(x) \neq COL(y) \end{cases}$$

Apply Large Ramsey theorem to get a Large Homog Set

$$\{h_1 < h_2 < \cdots < h_{h_1+1}\}$$

If COL'-homog color is EQ then H is homog for COL.
if COL'-homog color is NEQ then H is rainb for COL.

Midterm Question

Prove or Disprove:
$(\forall \mathrm{COL}\colon \mathsf{Q} \rightarrow [100])(\exists H \subseteq \mathsf{Q})[H \text{ Q-homog}]$

Midterm Question

Prove or Disprove:
$(\forall \mathrm{COL}\colon \mathsf{Q} \rightarrow [100])(\exists H \subseteq \mathsf{Q})[H \text{ Q-homog}]$
Its true. We prove it on next slide.

$(\forall c)(\forall \mathrm{COL}\colon \mathbf{Q} \to [c])(\exists H \subseteq \mathbf{Q})H$ is $\mathbf{Q}$-homog

We prove this by induction on c.

$(\forall c)(\forall \mathrm{COL}\colon \mathbf{Q} \to [c])(\exists H \subseteq \mathbf{Q})H$ is **Q**-homog

We prove this by induction on c.

IB $c = 1$. Obviously true.

$(\forall c)(\forall \mathrm{COL}\colon \mathbf{Q} \to [c])(\exists H \subseteq \mathbf{Q})H$ is Q-homog

We prove this by induction on c.

IB $c = 1$. Obviously true.

IH Assume true for $c - 1$.

$(\forall c)(\forall \mathrm{COL}\colon \mathbf{Q} \rightarrow [c])(\exists H \subseteq \mathbf{Q}) H$ is Q-homog

We prove this by induction on c.

IB $c = 1$. Obviously true.

IH Assume true for $c - 1$.

We will try to proof Thm at c. Either

- We succeed! YEAH!

$(\forall c)(\forall \mathrm{COL}\colon \mathbf{Q} \rightarrow [c])(\exists H \subseteq \mathbf{Q}) H$ is Q-homog

We prove this by induction on c.

IB $c = 1$. Obviously true.

IH Assume true for $c - 1$.

We will try to proof Thm at c. Either

- We succeed! YEAH!
- We fail! Then we will have an open interval where COL is never color c. Hence we have $\mathrm{COL}\colon \mathsf{Q} \rightarrow [c-1]$. Then use IH.

$\mathrm{COL}\colon \mathbf{Q} \to [c])(\exists H \subseteq \mathbf{Q})H$ is Q-homog

Let $\mathrm{COL}\colon \mathsf{Q} \to [c]$.

$\mathrm{COL}\colon \mathbf{Q} \rightarrow [c])(\exists H \subseteq \mathbf{Q}) H$ is Q-homog

Let $\mathrm{COL}\colon \mathsf{Q} \rightarrow [c]$.

We define a sequence $q_1, q_2, \ldots$ such that $\{q_1, q_2, \ldots\}$ is Q-homog OR we fail.

$\mathrm{COL}\colon \mathbf{Q} \to [c])(\exists H \subseteq \mathbf{Q})H$ is Q-homog

Let $\mathrm{COL}\colon \mathsf{Q} \to [c]$.

We define a sequence $q_1, q_2, \ldots$ such that $\{q_1, q_2, \ldots\}$ is Q-homog OR we fail.

Let $q_1 \in \mathsf{Q}$ such that $\mathrm{COL}(q_1) = c$.

$\mathrm{COL}\colon \mathbf{Q} \to [c])(\exists H \subseteq \mathbf{Q})H$ is Q-homog

Let $\mathrm{COL}\colon \mathsf{Q} \to [c]$.

We define a sequence $q_1, q_2, \ldots$ such that $\{q_1, q_2, \ldots\}$ is Q-homog OR we fail.

Let $q_1 \in \mathsf{Q}$ such that $\mathrm{COL}(q_1) = c$.

Assume $q_1, \ldots, q_n$ have been defined and are all color c. Order them to get $p_1 < \cdots < p_n$.

$\mathrm{COL}\colon \mathbf{Q} \rightarrow [c])(\exists H \subseteq \mathbf{Q})H$ is Q-homog

Let $\mathrm{COL}\colon \mathsf{Q} \rightarrow [c]$.
We define a sequence $q_1, q_2, \ldots$ such that $\{q_1, q_2, \ldots\}$ is Q-homog OR we fail.
Let $q_1 \in \mathsf{Q}$ such that $\mathrm{COL}(q_1) = c$.
Assume $q_1, \ldots, q_n$ have been defined and are all color c. Order them to get $p_1 < \cdots < p_n$.

- If $(\exists q < p_1)[\mathrm{COL}(q) = c]$ then let q_{n+1} be q.

$\mathrm{COL}\colon \mathbb{Q} \to [c])(\exists H \subseteq \mathbb{Q})H$ is Q-homog

Let $\mathrm{COL}\colon \mathsf{Q} \to [c]$.
We define a sequence $q_1, q_2, \ldots$ such that $\{q_1, q_2, \ldots\}$ is Q-homog OR we fail.
Let $q_1 \in \mathsf{Q}$ such that $\mathrm{COL}(q_1) = c$.
Assume $q_1, \ldots, q_n$ have been defined and are all color c. Order them to get $p_1 < \cdots < p_n$.

- If $(\exists q < p_1)[\mathrm{COL}(q) = c]$ then let q_{n+1} be q.
  If NOT then $\mathrm{COL}\colon (p_1 - 1, p_1) \to [c-1]$.
- For $1 \le i \le n$

$\mathrm{COL}\colon \mathbf{Q} \to [c])(\exists H \subseteq \mathbf{Q})H$ is Q-homog

Let $\mathrm{COL}\colon \mathsf{Q} \to [c]$.

We define a sequence $q_1, q_2, \ldots$ such that $\{q_1, q_2, \ldots\}$ is Q-homog OR we fail.

Let $q_1 \in \mathsf{Q}$ such that $\mathrm{COL}(q_1) = c$.

Assume $q_1, \ldots, q_n$ have been defined and are all color c. Order them to get $p_1 < \cdots < p_n$.

- If $(\exists q < p_1)[\mathrm{COL}(q) = c]$ then let q_{n+1} be q.
  If NOT then $\mathrm{COL}\colon (p_1 - 1, p_1) \to [c-1]$.
- For $1 \le i \le n$
  If $(\exists p_i < q < p_{i+1})[\mathrm{COL}(q) = c]$ then let q_{n+i+1} be q.

$\mathrm{COL}\colon \mathsf{Q} \to [c])(\exists H \subseteq \mathsf{Q})H$ is Q-homog

Let $\mathrm{COL}\colon \mathsf{Q} \to [c]$.

We define a sequence $q_1, q_2, \ldots$ such that $\{q_1, q_2, \ldots\}$ is Q-homog OR we fail.

Let $q_1 \in \mathsf{Q}$ such that $\mathrm{COL}(q_1) = c$.

Assume $q_1, \ldots, q_n$ have been defined and are all color c. Order them to get $p_1 < \cdots < p_n$.

- If $(\exists q < p_1)[\mathrm{COL}(q) = c]$ then let q_{n+1} be q.
  If NOT then $\mathrm{COL}\colon (p_1 - 1, p_1) \to [c-1]$.
- For $1 \le i \le n$
  If $(\exists p_i < q < p_{i+1})[\mathrm{COL}(q) = c]$ then let q_{n+i+1} be q.
  If NOT then $\mathrm{COL}\colon (p_i, p_{i+1}) \to [c-1]$.
- If $(\exists p_1 < q)[\mathrm{COL}(q) = c]$ then let q_{2n+2} be q.

$\mathrm{COL}\colon \mathsf{Q} \to [c])(\exists H \subseteq \mathsf{Q}) H$ is Q-homog

Let $\mathrm{COL}\colon \mathsf{Q} \to [c]$.
We define a sequence $q_1, q_2, \ldots$ such that $\{q_1, q_2, \ldots\}$ is Q-homog OR we fail.
Let $q_1 \in \mathsf{Q}$ such that $\mathrm{COL}(q_1) = c$.
Assume $q_1, \ldots, q_n$ have been defined and are all color c. Order them to get $p_1 < \cdots < p_n$.

- If $(\exists q < p_1)[\mathrm{COL}(q) = c]$ then let q_{n+1} be q.
  If NOT then $\mathrm{COL}\colon (p_1 - 1, p_1) \to [c-1]$.
- For $1 \le i \le n$
  If $(\exists p_i < q < p_{i+1})[\mathrm{COL}(q) = c]$ then let q_{n+i+1} be q.
  If NOT then $\mathrm{COL}\colon (p_i, p_{i+1}) \to [c-1]$.
- If $(\exists p_1 < q)[\mathrm{COL}(q) = c]$ then let q_{2n+2} be q.
  If NOT then $\mathrm{COL}\colon (p_n, p_n + 1) \to [c-1]$.

$\mathrm{COL}\colon \mathbf{Q} \rightarrow [c])(\exists H \subseteq \mathbf{Q})H$ is Q-homog

Let $\mathrm{COL}\colon \mathsf{Q} \rightarrow [c]$.

We define a sequence $q_1, q_2, \ldots$ such that $\{q_1, q_2, \ldots\}$ is Q-homog OR we fail.

Let $q_1 \in \mathsf{Q}$ such that $\mathrm{COL}(q_1) = c$.

Assume $q_1, \ldots, q_n$ have been defined and are all color c. Order them to get $p_1 < \cdots < p_n$.

- If $(\exists q < p_1)[\mathrm{COL}(q) = c]$ then let q_{n+1} be q.
  If NOT then $\mathrm{COL}\colon (p_1 - 1, p_1) \rightarrow [c-1]$.
- For $1 \le i \le n$
  If $(\exists p_i < q < p_{i+1})[\mathrm{COL}(q) = c]$ then let q_{n+i+1} be q.
  If NOT then $\mathrm{COL}\colon (p_i, p_{i+1}) \rightarrow [c-1]$.
- If $(\exists p_1 < q)[\mathrm{COL}(q) = c]$ then let q_{2n+2} be q.
  If NOT then $\mathrm{COL}\colon (p_n, p_n + 1) \rightarrow [c-1]$.

Case 1 $q_1, q_2, \ldots$ defined. It forms countable, dense, set with no endpoints.

$\mathrm{COL}\colon \mathsf{Q} \to [c])(\exists H \subseteq \mathsf{Q})H$ is Q-homog

Let $\mathrm{COL}\colon \mathsf{Q} \to [c]$.
We define a sequence $q_1, q_2, \ldots$ such that $\{q_1, q_2, \ldots\}$ is Q-homog OR we fail.
Let $q_1 \in \mathsf{Q}$ such that $\mathrm{COL}(q_1) = c$.
Assume $q_1, \ldots, q_n$ have been defined and are all color c. Order them to get $p_1 < \cdots < p_n$.

- If $(\exists q < p_1)[\mathrm{COL}(q) = c]$ then let q_{n+1} be q.
  If NOT then $\mathrm{COL}\colon (p_1 - 1, p_1) \to [c-1]$.
- For $1 \le i \le n$
  If $(\exists p_i < q < p_{i+1})[\mathrm{COL}(q) = c]$ then let q_{n+i+1} be q.
  If NOT then $\mathrm{COL}\colon (p_i, p_{i+1}) \to [c-1]$.
- If $(\exists p_1 < q)[\mathrm{COL}(q) = c]$ then let q_{2n+2} be q.
  If NOT then $\mathrm{COL}\colon (p_n, p_n + 1) \to [c-1]$.

Case 1 $q_1, q_2, \ldots$ defined. It forms countable, dense, set with no endpoints.

Case 2 Construction stops someplace. $\exists a < b$ such that $\mathrm{COL}\colon (a, b) \to [c-1]$. Induct.