Homework 06
Morally Due Tue March 29 at 3:30PM. Dead Cat March 31 at 3:30

IN THIS HW WHENEVER I SAY “A SET OF POINTS IN THE PLANE” I MEAN THAT THEY HAVE NO THREE COLINEAR.

1. (0 points) What is your name? Write it clearly. When is the take-home final due?

2. (35 points) Let $N(k)$ be the least $n$ such that for all sets of $n$ points there is a subset of $k$ of them that that form a convex $k$-gon.

We begin a proof that $N(k)$ exists and you need to finish it.

We show that $n = R_3(k)$ suffice. Let $X$ be a set of $n = R_3(k)$ points in the plane. Let the points be $p_1, p_2, \ldots, p_n$.

Color $(p_i, p_j, p_k)$ (with $i < j < k$) RED if $p_i, p_j, p_k$ is clockwise.

Color $(p_i, p_j, p_k)$ (with $i < j < k$) BLUE if $p_i, p_j, p_k$ is counter clockwise.

The Homogenous set of size $k$ is a convex $k$-gon because FILL THIS IN.
3. (35 points)

Def $\text{PH}(k)$ is the least $n \geq$ such that for all 2-colorings of $\binom{\{k,\ldots,n\}}{2}$ there exists a large homog set. WE ALSO REQUIRE $|\{k,\ldots,n\}| \geq 3$ SO THAT $\text{PH}(1)$ IS NOT TRIVIAL. THERE ARE ALSO OTHER REASONS.

Show that $\text{PH}(1) \leq 6$. 

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4. (30 points) Recall:

If \( n \equiv 1 \pmod{2} \) then for any COL: \( \binom{n}{2} \rightarrow [2] \) there exists at least

\[
\frac{n^3}{24} - \frac{n^2}{4} + \frac{5n}{24}
\]

monochromatic \( K_3 \)'s.

We will vary this in two ways.

(a) (15 points) Find a function \( f \) such that the followings is true:

If \( n \equiv 0 \pmod{2} \) then for any COL: \( \binom{n}{2} \rightarrow [n] \) there exists at least

\( f(n) \) monochromatic \( K_3 \)'s.

Prove your result.

(b) (15 points) We are interested in what happens if you have THREE colors. Do some empirical studies to try to find a function \( f \) such that the following holds:

If COL: \( \binom{n}{2} \rightarrow [3] \) then there exists at least \( f(n) \) monochromatic \( K_3 \)'s. \( (f(n) \) an be approximate. For example, if the problem was for 2-coloring then \( f(n) \) could be \( \frac{n^3}{24} \).)

(HINT: Use the code you wrote for the midterm; however, only use the case of \( p_1 = p_2 = p_3 = \frac{1}{3} \).)

(c) (Extra Credit, 0 points) PROVE a result along the lines of:

If \( n \) satisfies condition YOU FILL IN and COL: \( \binom{n}{2} \rightarrow [3] \) then there exists at least \( f(n) \) monochromatic \( K_3 \)'s.)

(HINT: Use the code you wrote for the midterm; however, only use the case of \( p_1 = p_2 = p_3 = \frac{1}{3} \).)
5. (Extra Credit, but THINK ABOUT IT. WARNING- I have not done this problem)

Let $X$ be an infinite set of points $p_1, p_2, p_3, \ldots$. Let $\text{COLN} \rightarrow \omega$ be defined as follows:

$$\text{COL}(i, j, k) = \text{the number of points inside the } (i, j, k) \text{ triangle.}$$

Apply the 3-ary Can Ramsey Theorem to this Coloring. NOW WHAT?

6. (Extra Credit, but THINK ABOUT IT–WARNING: the way I know how to do this is based on material you have not seen) We want to write a sentence $\phi$ in the language of graphs such that $G \models \phi$ IFF $G$ has an even number of vertices.
Is this possible?