Homework 08
Morally Due Tue April 5 at 3:30PM. Dead Cat April 7 at 3:30
WARNING: THE HW IS THREE PAGES LONG

1. (0 points) What is your name? Write it clearly. When is the take-home final due?

2. (35 points) Give a sentence ϕ in the language of graphs such that

$$\text{spec}(\phi) = \{n : n \equiv 1 \pmod 4\}.$$

SOLUTION

We want to say that there is one isolated point, and aside from that all of the points come in sets of C_4's.

For ease of notation you an write things like $(\forall x, x \neq y)$.

$(\exists x)[$

the AND of the following:

- $(\forall y)[\neg E(x,y)]$. x is an isolated vertex.
- $(\forall y \neq x)(\exists z_1, z_2)[E(y,z_1) \wedge E(y,z_2) \wedge (\forall w \neq z_1, z_2)[\neg E(y,w)]]$
  All vertices except x have degree exactly 2.
- $(\forall y \neq x)(\exists y_1, y_2, y_3)[E(y,y_1) \wedge E(y_1,y_2) \wedge E(y_2,y_3) \wedge E(y_3,y)]$
  Every vertex except x is a member of a C_4. Note that since all such vertices have degree 2, the y_1, y_2, y_3, y are in a C_4 and are not connected to anything else.)

$]$.

END OF SOLUTION

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3. (35 points) For this problem we are use the language of 3-hypergraphs. So there is only one predicate: $E(x,y,z)$. (We assume E is symmetric so

$$E(x,y,z) = E(x,z,y) = E(y,x,z) = E(y,z,x) = E(z,x,y) = E(z,y,z).$$

)

Let ϕ be a sentence in this language of the form

$$\phi = (\exists x_1) \cdots (\exists x_n)(\forall y_1) \cdots (\forall y_m)[\psi(x_1, \ldots, x_n, y_1, \ldots, y_m)]$$

Fill in the blank in the following theorem and proof the theorem. Make it VERY CLEAR what your XXX is. (The TAs get annoyed if they have to search for it. They also get annoyed when I ask them to search for $R(5)$.)

If $(\exists N \geq XXX(n,m))[N \in \text{spec}(\phi)]$ then

$$\{n+m, n+m+1, \ldots\} \subseteq \text{spec}(\phi).$$

SOLUTION

We determine $XXX(n,m)$ later. We denote it XXX.

Assume there is a 3-hypergraph G on $\ge XXX$ vertices such that ϕ holds. Let $u_1,\ldots,u_n$ be the witnesses. Let

$$U = \{u_1,\ldots,u_n\}.$$

Then the remaining vertices are X with $X = XXX - n$. Call these vertices Let $ZZZ = XXX - n$. We determine ZZZ later.

$$Y = \{y_1,\ldots,y_{ZZZ-n}\}$$

We want to make all of the y_i look the same to all elements of U.

Map each $y_i \in Y$ to the following $\binom{n}{2}$ sized vector. Index the vector by $\binom{[n]}{2}$.

The $\{a,b\}$ entry is $E(y_i,a,b)$.

This is a mapping of ZZZ elements to $2^{\binom{n}{2}}$ elements. Hence some element of the range is mapped to $\frac{ZZZ}{2^{\binom{n}{2}}}$ times.

Let $ZZZ = 2^{\binom{n}{2}}WWW$. We determine WWW later.

SO there are now $\{x_1,\ldots,x_{WWW}\}$ that all have the same relation to all $u \in U$.

We now want all of the x_i's to have the same relation to each other. Hence we will be using 3-ary Ramsey. Let $WWW = R_3(m)$.

FINAL

$$XXX = ZZZ + n = 2^{\binom{n}{2}}WWW + n = 2^{\binom{n}{2}}R_3(m).$$

From this point the proof is similar to what I did in class.

END OF SOLUTION

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4. (30 points) (This problem is inspired by my talk on my book.)

   A number of the form $x^2 + x$ where $x \in \mathsf{N}$, $x \geq 1$, is called a *Liam*. The first few Liam's are $2, 6, 12, 20, 30, 42, 56, 72, 90$.

   Let $L(c)$ be the least n (if it exists) so that for all c-colorings of $\{1, \ldots, n\}$ there exists two numbers that are the same color that are a Liam apart.

   (a) Find an upper bound on $L(2)$.

   (b) Find an upper bound on $L(3)$.

SOLUTION

a) Let $\mathrm{COL}\colon [n] \rightarrow [2]$. We determine n later.

Assume $\mathrm{COL}(1) = 1$. Since 2 is Liam we have that

$\mathrm{COL}(1) = \mathrm{COL}(5) = \mathrm{COL}(9) = \mathrm{COL}(13)$

But 1 and 13 are 12 apart, and 12 is a Liam. Hence $L(2) \le 13$.

b) Let $\mathrm{COL}\colon [n] \rightarrow [3]$. We determine n later.

We note the following relationship among Liam numbers: $30 + 12 = 42$.

Assume that $\mathrm{COL}(1) = 1$.

Since $\mathrm{COL}(1) \ne \mathrm{COL}(1 + 12)$ we can take $\mathrm{COL}(1 + 12) = 2$.

Since $\mathrm{COL}(1) \ne \mathrm{COL}(1+42)$ and $\mathrm{COL}(1+12) \ne \mathrm{COL}(1+42)$, we can take $\mathrm{COL}(1 + 42) = 3$.

Now look at $\mathrm{COL}(1 + 54)$

Since $1 + 54 = (1 + 12) + 42$, $\mathrm{COL}(1 + 54) \ne \mathrm{COL}(1 + 12) = 2$.

Since $1 + 54 = (1 + 42) + 12$, $\mathrm{COL}(1 + 54) \ne \mathrm{COL}(1 + 42) = 3$.

Hence $\mathrm{COL}(1 + 54) = 1$.

More succinctly: $\mathrm{COL}(1) = \mathrm{COL}(1 + 54) = \mathrm{COL}(1 + 2 \times 54) = \cdots = \mathrm{COL}(1 + 54k)$.

So we need a value of k such that $54k$ is Liam.

$$54k = x^2 + x = x(x + 1)$$

OH- lets take $x = 27$.

$$54k = 27 \times 28 = 54 \times 14$$

Great, we take $k = 14$.

$54 \times 14 = 27 \times 28$ is Liam.

$54 \times 14 = 756$.

SO $L(3) \le 757$.

I suspect we can do much better.

END OF SOLUTION