HW08 Solutions

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Every non-x vert is in a C_4. All non-x verts have deg 2, so the y_1, y_2, y_3, y are in a C_4 and are not connected to anything else.

Problem 3

We use the language of 3-hypergraphs. One predicate: $E(x, y, z)$.
We assume E is symmetric.

$$\phi = (\exists x_1) \cdots (\exists x_n)(\forall y_1) \cdots (\forall y_m)[\psi(x_1, \ldots, x_n, y_1, \ldots, y_m)]$$

If $(\exists N \ge XXX(n, m))[N \in \mathrm{spec}(\phi)]$ then

$$\{n + m, n + m + 1, \ldots\} \subseteq \mathrm{spec}(\phi).$$

Fill in the XXX and prove it.

SOLUTION to Problem 3

Assume $\exists$ 3-hypergraph $G = (V, E)$ on $\geq XXX$ vertices, $G \models \phi$.
Witnesses: $u_1, \ldots, u_n$ be the witnesses.

$$U = \{u_1, \ldots, u_n\} \qquad Y = V - U \qquad |Y| = XXX - n = ZZZ.$$

$$Y = \{y_1, \ldots, y_{ZZZ}\}$$

Want Y superhomog.
Map $y_i \in Y$ to the following $\binom{n}{2}$ sized vector. Index the vector by $\binom{[n]}{2}$.
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SOLUTION to Problem 3 (cont)

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$ZZZ = XXX - n$

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So what is XXX?

$$XXX = ZZZ + n = 2^{\binom{n}{2}} WWW + n = 2^{\binom{n}{2}} R_3(m) + n.$$

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1. Find an upper bound on $L(2)$.
2. Find an upper bound on $L(3)$.

SOLUTION to *a*

We show that $(\forall \mathrm{COL}\colon [13] \rightarrow [2])$ there exists x, y a Liam apart that are the same color.

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Contradiction.

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SOLUTION to *b* (Diagram)

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More generally, $\mathrm{COL}(x) = \mathrm{COL}(x+54)$.

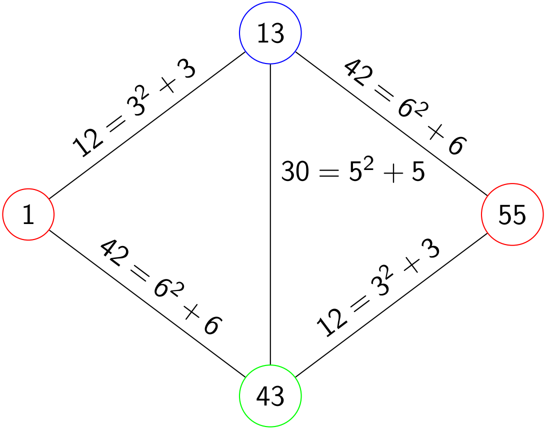

Figure: $\mathrm{COL}(x) = \mathrm{COL}(x+54)$

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I suspect we can do much better.