Homework DS

Definition Let $G = (V, E)$ be a graph. $D \subseteq V$ is a *dominating set* if

$$(\forall v \in V)[v \in D \vee (\exists y \in D)[(x, y) \in E].$$

Note that every graph has a dominating set of size n, just take $D = V$. Does every graph have a SMALLER dominating set? How much smaller? In this problem we will sketch a proof of the following:

There exists a function α such that,

a) For all $d \in \mathsf{N}$, $0 < \alpha(d) < 1$.

b) The function $\alpha(d)$ is DECREASING.

c) For every graph with min degree $\geq d$ there is a dominating set of size $\leq \alpha(d)n$.

YOU will fill in the details including the function α.

Thm If $G = (V, E)$ is a graph on n vertices with min degree $\geq d$ then G has a dominating set of size $\leq \alpha(n)d$.

Pf Let p be a probability to be determined by YOU later.

Pick $X \subseteq V$ as follows: For every $v \in V$ choose v with probability p.

QUESTION 1 What is $E(|X|)$? It will be a function of n, p.

Let $Y \subseteq V - X$ be the vertices that DO NOT have an edge to an element of X. Formally

$$Y = \{y \in V - X : (\forall x \in X)[(x, y) \notin E].$$

QUESTION 2 Give an upper bound on $E(|Y|)$. It will be a function of n, d, p.

Note that $X \cup Y$ is a dominating set. We want to pick p so that $|X \cup Y|$ is small.

QUESTION 3 What is $E(|X \cup Y|)$? (Hint: This is very easy by the linearity of expectation.)

QUESTION 4 Pick p to make $E(|X \cup Y|)$ smaller than n (Hint: Find an upper bound on $E(|X \cup Y|)$ and minimize that bound. Use that $(1-p) \leq e^{-p}$.)

QUESTION 5 State and proof a theorem of the form:

Thm If $G = (V, E)$ is a graph on n vertices with min degree $\geq d$ then G has a dominating set of size $\leq \alpha(d)n$.