Take Home Midterm

Morally Due Tue Mar 15 at 3:30PM. Dead Cat Mar 17 at 3:30

1. (0 points) What is your name? Write it clearly.

2. (25 points) Prove the following and fill in the $f(k)$.

   Theorem For all k there exists $n = f(k)$ such that the following holds.

   For all pairs of colorings:

   $\mathrm{COL}_1\colon \binom{[n]}{1} \to [2]$,

   $\mathrm{COL}_2\colon \binom{[n]}{2} \to [2]$

   there exists $H \subseteq [n]$ and colors $c_1, c_2 \in \{1, 2\}$ (it's okay if $c_1 = c_2$) such that

   - H is of size k,
   - every element of H is colored c_1, and
   - every element of $\binom{H}{2}$ is colored c_2.

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3. (25 points) Let T be the set of trees and $\preceq$ be the minor ordering. Show that $(T, \preceq)$ is a wqo.

   You may use any theorem that was PROVEN in class or on the HW. (Note that we DID NOT prove the Graph Minor Theorem, so you can't use that.)

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4. (25 points) Let Q be the rationals. PROVE or DISPROVE:

   For every COL: $\mathsf{Q} \rightarrow [100]$ *there exists an* $H \subseteq \mathsf{Q}$ *and a color* c *such that*

   - *H has the same order type as the rationals (so H is a countable set without endpoints where between any two elements is an element), and*
   - *every number in H is the same color.*

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5. (25 points)

ONE (0 points but you need to do this for later parts) Write a program such that:

- *Input* is $n \in \mathsf{N}$ and $0 \le p_1, p_2, p_3 \le 1$ with $p_1 + p_2 + p_3 = 1$ and $p_1 \le p_2 \le p_3$.
- *Output* is a COL: $\binom{[n]}{2} \to [3]$ that is generated randomly with each edge being colored 1 with prob p_1, 2 with prob p_2, and 3 with prob p_3.

TWO (0 points but you need to do it for later parts) Write a program that will, given COL: $\binom{[n]}{2} \to [3]$, counts how many monochromatic triangles it has.

THREE (0 points but you need this for later parts) Write a program that does the following (I use psuedocode.)

(a) Input n. (n will be ≥ 6.)

For $p_1, p_2, p_3 \in \{0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8\}$ such that $p_1 + p_2 + p_3 = 1$ and $p_1 \le p_2 \le p_2$.

$L = \binom{n}{3}$ (the number of triangles in K_n).
$n_0 = 0$, $n_1 = 0$, ..., $n_L = 0$.
(n_i will be the number of colorings that have i mono triangles. Initially this is 0.)

i. For $i = 1$ to 1000

Randomly color the edges of K_n by coloring 1 with prob p_1, 2 with prob p_2, 3 with prob p_3.

A. Find j, the number of mono triangles.

B. $n_j = n_j + 1$.

ii. $n_{\max} = \max\{n_0, \ldots, n_L\}$.

iii. $j_{\max}$ is the j such that $n_j = n_{\max}$. (If there is more than one j, which is unlikely, take the least one.)

FOUR (20 points) Use your program to produce tables of data. Our interest is in which n_j's are always 0 and which n_j occurs the most often. The tables should look like what is below except that I made up the answers.

$n = 5$

p_1	p_2	p_3	$\{j : n_j = 0\}$	$j_{\max}$	$n_{\max}$
0.1	0.1	0.8	$\{3, 8\}$	3	109
0.1	0.2	0.7	$\{1, 9\}$	7	108
0.1	0.3	0.6	$\{2, 4, 8\}$	1	200
0.1	0.4	0.5	$\{2\}$	1	10
0.2	0.2	0.6	$\{1, 2, 4\}$	3	300
0.2	0.3	0.5	$\{3, 4, 5, 7\}$	2	401
0.2	0.4	0.4	$\{1, 2, 9\}$	10	512
0.3	0.3	0.4	$\{2, 3, 8\}$	7	70

$n = 6$ SIMILAR TO ABOVE

$n = 7$ SIMILAR TO ABOVE

$\vdots$

$n = 20$ SIMILAR TO ABOVE

FIVE (0 points) Looking at the data formulate a conjecture about colorings of K_n.

Extra Credit Prove your conjecture.