There is no 3-coloring of 15×15

Rectangle Free Sets

Assume there is a 3-coloring of $G_{15,15}$.

$$15 \times 15 = 225$$

Rectangle Free Sets

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There is a rectangle free set X, $|X| \geq \frac{225}{3} = 75$.

Our Plan

For $1 \leq i \leq 15$
let x_i be the number of elements of X in the ith column.

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$$\sum_{i=1}^{15} \binom{x_i}{2}.$$

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Plan The number of pairs of $\{1, \ldots, 15\}$ is $\binom{15}{2} = 105$.
We will find a lower bound L on $\sum_{i=1}^{15} \binom{x_i}{2}$.

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Plan The number of pairs of $\{1, \ldots, 15\}$ is $\binom{15}{2} = 105$.
We will find a lower bound L on $\sum_{i=1}^{15} \binom{x_i}{2}$.

We will show $L > 105$, hence some four elements of X form a rectangle.

Inequality

Want to show that $\sum_{i=1}^{15} \binom{x_i}{2} \geq 106$.

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Want to show that $\sum_{i=1}^{15} \binom{x_i}{2} \geq 106$.

Want to find MIN of

$$\sum_{i=1}^{15} \binom{x_i}{2}$$

relative to the constraint

$$\sum_{i=1}^{15} x_i = 75.$$

Well Known Theorem

Over the REALS:
$\sum_{i=1}^{15} \frac{x_i(x_i-1)}{2}$
relative to the constraint
$\sum_{i=1}^{15} x_i = 75.$
is MINIMIZED if all of the x_is are equal.

We take $x_i = 75/15 = 5$.

$$\sum_{i=1}^{15} \frac{x_i(x_i-1)}{2} \geq \sum_{i=1}^{15} \frac{5 \times 4}{2} = 15 \times 10 = 150.$$

Recap and Finish

The number of vertical pairs is $\binom{15}{2} = 105$

The number of vertical pairs of points in X is

$$\geq \sum_{i=1}^{15} \frac{x_i(x_i - 1)}{2} \geq \sum_{i=1}^{15} \frac{5 \times 4}{2} = 15 \times 10 = 150.$$

Recap and Finish

The number of vertical pairs is $\binom{15}{2} = 105$

The number of vertical pairs of points in X is

$$\geq \sum_{i=1}^{15} \frac{x_i(x_i - 1)}{2} \geq \sum_{i=1}^{15} \frac{5 \times 4}{2} = 15 \times 10 = 150.$$

Hence some vertical pair of points occurs twice, so X is not rectangle free.

There is no 3-coloring of 14×14

Rectangle Free Sets

Assume there is a 3-coloring of $G_{14,14}$.

$$14 \times 14 = 196$$

Rectangle Free Sets

Assume there is a 3-coloring of $G_{14,14}$.

$$14 \times 14 = 196$$

There is a rectangle free set X, $|X| \geq \left\lceil \frac{196}{3} \right\rceil = 66$.

Rectangle Free Sets

Assume there is a 3-coloring of $G_{14,14}$.

$$14 \times 14 = 196$$

There is a rectangle free set X, $|X| \geq \lceil \frac{196}{3} \rceil = 66$.

For $1 \leq i \leq 14$
let x_i be the number of elements of X in the ith column.
Need

$$\sum_{i=1}^{14} \binom{x_i}{2} \geq \binom{14}{2} = 91.$$

MIN the sum

Over the REALS:
$\sum_{i=1}^{14} \frac{x_i(x_i-1)}{2}$
relative to the constraint
$\sum_{i=1}^{14} x_i = 66.$
is MINIMIZED if all of the x_i's are equal.

We take $x_i = \frac{66}{14} = \frac{33}{7}$.
We note that $\frac{1}{2} \times \frac{33}{7}(\frac{33}{7} - 1) = \frac{429}{49}$

$$\sum_{i=1}^{14} \frac{x_i(x_i-1)}{2} \ge \sum_{i=1}^{14} \frac{429}{49} = \frac{858}{7} = 122+ > 91$$

DONE.

There is no 3-coloring of 13×13

Rectangle Free Sets

Assume there is a 3-coloring of $G_{13,13}$.

$$13 \times 13 = 169$$

Rectangle Free Sets

Assume there is a 3-coloring of $G_{13,13}$.

$$13 \times 13 = 169$$

There is a rectangle free set X, $|X| \geq \left\lceil \frac{169}{3} \right\rceil = 57$.

Rectangle Free Sets

Assume there is a 3-coloring of $G_{13,13}$.

$$13 \times 13 = 169$$

There is a rectangle free set X, $|X| \geq \lceil \frac{169}{3} \rceil = 57$.

For $1 \leq i \leq 13$
let x_i be the number of elements of X in the ith column.
Need

$$\sum_{i=1}^{13} \binom{x_i}{2} \geq \binom{13}{2} = 78.$$

MIN the sum

Over the REALS:
$\sum_{i=1}^{13} \frac{x_i(x_i-1)}{2}$
relative to the constraint
$\sum_{i=1}^{13} x_i = 57.$
is MINIMIZED if all of the x_i's are equal.

We take $x_i = \frac{57}{13}$
We note that $\frac{1}{2} \times \frac{57}{13}(\frac{57}{13} - 1) = \frac{1254}{169}$

$$\sum_{i=1}^{13} \frac{x_i(x_i-1)}{2} \geq 13 \times \frac{1254}{169} = 96+ > 78$$

DONE.

There is no 3-coloring of 12×12

Rectangle Free Sets

Assume there is a 3-coloring of $G_{12,12}$.

$$12 \times 12 = 144$$

Rectangle Free Sets

Assume there is a 3-coloring of $G_{12,12}$.

$$12 \times 12 = 144$$

There is a rectangle free set X, $|X| \geq \lceil \frac{144}{3} \rceil = 48$.

Rectangle Free Sets

Assume there is a 3-coloring of $G_{12,12}$.

$$12 \times 12 = 144$$

There is a rectangle free set X, $|X| \geq \lceil \frac{144}{3} \rceil = 48$.

For $1 \leq i \leq 12$
let x_i be the number of elements of X in the ith column.
Need

$$\sum_{i=1}^{12} \binom{x_i}{2} \geq \binom{12}{2} = 66.$$

MIN the sum

Over the REALS:
$\sum_{i=1}^{12} \frac{x_i(x_i-1)}{2}$
relative to the constraint
$\sum_{i=1}^{12} x_i = 48.$
is MINIMIZED if all of the x_i's are equal.

We take $x_i = \frac{48}{12} = 4$
We note that $\frac{1}{2} \times 4 \times 3 = 6.$

$$\sum_{i=1}^{12} \frac{x_i(x_i - 1)}{2} \geq 12 \times 6 = 72 > 66$$

DONE.

There is no 3-coloring of 11×11

Rectangle Free Sets

Assume there is a 3-coloring of $G_{11,11}$.

$$11 \times 11 = 121$$

Rectangle Free Sets

Assume there is a 3-coloring of $G_{11,11}$.

$$11 \times 11 = 121$$

There is a rectangle free set X, $|X| \geq \lceil \frac{121}{3} \rceil = 41$.

Rectangle Free Sets

Assume there is a 3-coloring of $G_{11,11}$.

$$11 \times 11 = 121$$

There is a rectangle free set X, $|X| \geq \lceil \frac{121}{3} \rceil = 41$.

For $1 \leq i \leq 11$
let x_i be the number of elements of X in the ith column.
Need

$$\sum_{i=1}^{11} \binom{x_i}{2} \geq \binom{11}{2} = 55.$$

MIN the sum

Over the REALS:
$\sum_{i=1}^{11} \frac{x_i(x_i-1)}{2}$
relative to the constraint
$\sum_{i=1}^{11} x_i = 41.$
is MINIMIZED if all of the x_i's are equal.

We take $x_i = \frac{41}{11}$
We note that $\frac{1}{2} \times \frac{41}{11}(\frac{41}{11} - 1) = \frac{615}{121}$.

$$\sum_{i=1}^{11} \frac{x_i(x_i - 1)}{2} \geq 11\frac{615}{121} = 55+ > 55$$

DONE.

There is no 3-coloring of 10×10

Rectangle Free Sets

Assume there is a 3-coloring of $G_{10,10}$.

$$10 \times 10 = 100$$

Rectangle Free Sets

Assume there is a 3-coloring of $G_{10,10}$.

$$10 \times 10 = 100$$

There is a rectangle free set X, $|X| \geq \lceil \frac{100}{3} \rceil = 34$.

Rectangle Free Sets

Assume there is a 3-coloring of $G_{10,10}$.

$$10 \times 10 = 100$$

There is a rectangle free set X, $|X| \geq \lceil \frac{100}{3} \rceil = 34$.

For $1 \leq i \leq 10$
let x_i be the number of elements of X in the ith column.
Need

$$\sum_{i=1}^{10} \binom{x_i}{2} \geq \binom{10}{2} = 45.$$

MIN the sum

Over the REALS:
$\sum_{i=1}^{10} \frac{x_i(x_i-1)}{2}$
relative to the constraint
$\sum_{i=1}^{10} x_i = 34.$
is MINIMIZED if all of the x_i's are equal.

We take $x_i = \frac{34}{10} = \frac{17}{5}$
We note that $\frac{1}{2} \times \frac{17}{5}(\frac{17}{5} - 1) = \frac{102}{25}$.

$$\sum_{i=1}^{10} \frac{x_i(x_i-1)}{2} \ge 10\frac{102}{25} = 41+ > 45$$

THAT LAST LINE IS FALSE! So DO NOT have Proof that $G_{10,10}$ is NOT 3-col.