One Triangle, Two Triangles

William Gasarch
The following is the first theorem in Ramsey Theory:
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**Thm** For all 2-col of the edges of $K_6$ there is a mono $K_3$. 
**Thm** For all 2-cols of edges of $K_{12}$ there are 2 mono $K_3$'s

**Question** Find $n$ such that

1. For all 2-col of the edges of $K_n$ there are 2 mono $K_3$'s
2. There exists a 2-col of the edges of $K_{n-1}$ that does not have 2 mono $K_3$'s.
**Thm** For all 2-cols of edges of $K_{12}$ there are 2 mono $K_3$’s

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**VOTE** (1) $n = 12$,
**Thm** For all 2-cols of edges of $K_{12}$ there are 2 mono $K_3$’s

**Question** Find $n$ such that

1. For all 2-col of the edges of $K_n$ there are 2 mono $K_3$’s
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**VOTE** (1) $n = 12$, (2) $9 \leq n \leq 10$, 

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**VOTE** (1) $n = 12$, (2) $9 \leq n \leq 10$, (3) $6 \leq n \leq 8$. 
Thm  For all 2-cols of edges of $K_{12}$ there are 2 mono $K_3$'s.

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VOTE  (1) $n = 12$, (2) $9 \leq n \leq 10$, (3) $6 \leq n \leq 8$.

Answer  $n = 6$. 
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1. For all 2-col of the edges of $K_6$ there are 2 mono $K_3$'s
2. There exists a 2-col of the edges of $K_5$ that does not have any mono $K_3$'s.
**Thm** For all 2-cols of edges of $K_6$ there are 2 mono $K_3$'s

**Proof** Let $COL$ be a 2-col of the edges of $K_6$.

Let $R$, $B$, $M$, be the SET of RED, BLUE, and MIXED triangles.

$$|R| + |B| + |M| = \binom{6}{3} = 20.$$

We show that $|M| \leq 18$, so $|R| + |B| \geq 2$. 
A Mixed Triangle Has a Vertex Such That

\[ (v_2, v_1) \text{ is red, } (v_2, v_3) \text{ is blue. View this as } (v_2, \{v_1, v_3\}). \]

\[ (v_3, v_1) \text{ is red, } (v_3, v_2) \text{ is blue. View this as } (v_3, \{v_1, v_2\}). \]
Def A Zan is an element $(v, \{u, w\}) \in V \times \binom{V}{2}$ such that $v \not\in \{u, w\}$ and $\text{COL}(v, u) \neq \text{COL}(v, w)$. ZAN is the set of Zan’s.
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Map ZAN to \(M\) by mapping \((v, \{u, w\})\) to triangle \(\{v, u, w\}\).
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Claim This mapping is exactly 2-to-1.
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What Zan’s map to the triangle:
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What Zan’s map to the triangle:

\[(v_2, \{v_1, v_3\})\text{ and } (v_3, \{v_1, v_2\}).\]
Upper Bound on $M$

There is a 2-to-1 map from $ZAN$ to $M$. Hence

$$|M| = |ZAN|/2$$
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Now we want to bound $|ZAN|$. 

---

Look at vertex $v$. How many $ZAN$s use $v$ as their base point?

Depends on $\text{deg}_R(v)$ and $\text{deg}_B(v)$.

Thought experiment: If $\text{deg}_R(v) = 3$ and $\text{deg}_B(v) = 2$ then how many $ZAN$s are of the form $\{v, \{x, y\}\}$? $x$: $\text{COL}(v, x) = \text{RED}$. There are $\text{deg}_R(v)$ of them. $y$: $\text{COL}(v, y) = \text{BLUE}$. There are $\text{deg}_B(v)$ of them. So $v$ contributes $\text{deg}_R(v) \times \text{deg}_B(v)$.
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So $v$ contributes $\deg_R(v) \times \deg_B(v)$. 
Contributions!

Cases

1. $v$ has $\deg_R(v) = 5$ or $\deg_B(v) = 0$: $v$ contributes 0.

2. $v$ has $\deg_R(v) = 4$ or $\deg_B(v) = 1$: $v$ contributes 4.

3. $v$ has $\deg_R(v) = 3$ or $\deg_B(v) = 2$: $v$ contributes 6.

Max. 6 vertices, each contribute $\leq 6$, so $|M| = |ZAN|/2 \leq 6 \times 6/2 = 18$, so $|R| + |B| \geq 20 - |M| \geq 2$. 
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ZAN is max when each vertex: 3 R and 2 B (or 2 R and 3 B).

\[ |ZAN| \leq 6 \times 6 = 36. \]
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So there are at least 2 Mono Triangles.
Generalization

If we 2-color the edges of $K_n$ how many mono $K_3$'s do we have?
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If we 2-color the edges of $K_n$ how many mono $K_3$’s do we have?

VOTE (1) $\sim n^c$ for some $c < 1$, (2) $\sim n$ (3) $\sim n^2$, (4) $\sim n^3$. 
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**Answer** $\sim n^3$. Actually $\frac{n^3}{24} - \frac{n^2}{4} + O(n)$. 
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We do one case: $n \equiv 1 \pmod{2}$.
Let $COL$ be a coloring of the edges of $K_n$. 

If we 2-color the edges of $K_n$ how many mono $K_3$’s do we have?

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Then degree of each vertex is $n - 1 \equiv 0 \pmod{2}$.
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We do one case: $n \equiv 1 \pmod{2}$.
Let $COL$ be a coloring of the edges of $K_n$.
Then degree of each vertex is $n - 1 \equiv 0 \pmod{2}$.

We find an upper bound on $|ZAN|$.
Maximize $|ZAN|$

To maximize $|ZAN|$ we would, at each vertex, color half of the edges RED and half BLUE.
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$$|ZAN| \leq n \frac{(n-1)^2}{4} = \frac{(n-1)^2 n}{4} \text{ so}$$
Maximize |ZAN|

To maximize |ZAN| we would, at each vertex, color half of the edges RED and half BLUE. Each vertex contributes \( \left( \frac{n-1}{2} \right)^2 \) (this is in \( \mathbb{N} \) since \( n - 1 \equiv 0 \) (mod 2)).

\[
|ZAN| \leq n \frac{(n-1)^2}{4} = \frac{(n-1)^2 n}{4}
\]

so

\[
|M| = \frac{|ZAN|}{2} \leq \frac{(n-1)^2 n}{8}
\]
Recap

$$|M| \leq \frac{(n - 1)^2 n}{8}$$
Finishing Up The Proof

Recap

\[ |M| \leq \frac{(n - 1)^2 n}{8} \]

Recall

\[ |R| + |B| + |M| = \binom{n}{3} = \frac{n(n - 1)(n - 2)}{6} \]  hence
Finishing Up The Proof

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\[ |R| + |B| = \frac{n(n - 1)(n - 2)}{6} - |M| \]

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Finishing Up The Proof

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\[ |R| + |B| = \frac{n(n-1)(n-2)}{6} - |M| \quad \text{hence} \]

\[ |R| + |B| \geq \frac{n(n-1)(n-2)}{6} - \frac{(n-1)^2 n}{8} \]
Finishing Up The Proof

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\[ |M| \leq \frac{(n - 1)^2 n}{8} \]

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hence

\[ |R| + |B| \geq \frac{n(n-1)(n-2)}{6} - \frac{(n-1)^2 n}{8} = \frac{n^3}{24} - \frac{n^2}{4} + \frac{5n}{24} \]
What About The Other Cases?

We leave the other cases to the reader to both determine the theorem and prove it.
Can This Be Improved?

The bound is known to be tight.
The following is an early theorem in Ramsey Theory:

\[ \text{Thm} \]

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The following is an early theorem in Ramsey Theory:

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**Thm** For all 2-cols of edges of $K_{36}$ there are 2 mono $K_4$’s
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Smallest  $n$ such that $\forall$ 2-col of edges of $K_n \exists$ 2 mono $K_4$'s?
Trivial Theorem, Non Trivial Extension

**Thm** For all 2-cols of edges of $K_{36}$ there are 2 mono $K_4$'s

**Smallest** $n$ such that $\forall$ 2-col of edges of $K_n \exists$ 2 mono $K_4$'s?

**VOTE** (1) $n = 36$, 

(2) Some $n$, $19 \leq n \leq 35$, 

(3) $n = 18$. 

Answer This is really two questions.

1. As posed the answer is $n = 18$. Piwakowski and Radziszowski show that for every 2-col of $K_{18}$ there are 9 mono $K_4$'s. The proof used a clever computer search. This is interesting to know but the proof is not math-interesting.

2. I will present a math-interesting proof of the following: For all 2-cols of $K_{19}$ there are TWO mono $K_4$'s.
**Thm** For all 2-cols of edges of $K_{36}$ there are 2 mono $K_4$’s

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Thm For all 2-cols of edges of $K_{36}$ there are 2 mono $K_4$'s

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Thm For all 2-cols of edges of $K_{36}$ there are 2 mono $K_4$'s

Smallest $n$ such that $\forall$ 2-col of edges of $K_n$ $\exists$ 2 mono $K_4$'s?

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Trivial Theorem, Non Trivial Extension

**Thm** For all 2-cols of edges of $K_{36}$ there are 2 mono $K_4$'s

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**Answer** This is really two questions.

1. As posed the answer is $n = 18$. Piwakoswki and Radziszowski [https://www.cs.rit.edu/~spr/PUBL/paper40.pdf](https://www.cs.rit.edu/~spr/PUBL/paper40.pdf) showed that for every 2-col of $K_{18}$ there are 9 mono $K_4$’s. The proof used a clever computer search. This is interesting to know but the proof is not math-interesting.

2. I will present a math-interesting proof of the following: *For all 2-cols of $K_{19}$ there are TWO mono $K_4$’s.*
Proof of $K_{19}$ Two $K_4$ Theorem

**Thm** For all 2-cols of edges of $K_{19}$ there are 2 mono $K_4$'s
Proof of $K_{19}$ Two $K_4$ Theorem

**Thm** For all 2-cols of edges of $K_{19}$ there are 2 mono $K_4$’s
Assume you have a 2-col of the edges of $K_{19}$. 

**YOU’VE BEEN PUNKED** It is quite possible that the mono $K_4$ from $A_3$ and the mono $K_4$ from $A_9$ are the same.

For the real proof, see next slide.
**Proof of $K_{19}$ Two $K_4$ Theorem**

**Thm** For all 2-cols of edges of $K_{19}$ there are 2 mono $K_4$’s.

Assume you have a 2-col of the edges of $K_{19}$.

List out all subsets of $V = \{1, \ldots, 19\}$ of size $R(4) = 18$.

YOU'VE BEEN PUNKED. It is quite possible that the mono $K_4$ from $A_3$ and the mono $K_4$ from $A_9$ are the same.

For the real proof, see next slide.
Proof of $K_{19}$ Two $K_4$ Theorem

**Thm** For all 2-cols of edges of $K_{19}$ there are 2 mono $K_4$’s

Assume you have a 2-col of the edges of $K_{19}$.

List out all subsets of $V = \{1, \ldots, 19\}$ of size $R(4) = 18$.

$$A_1, A_2, \ldots, A_{19 \choose 18}.$$
Thm For all 2-cols of edges of $K_{19}$ there are 2 mono $K_4$’s.
Assume you have a 2-col of the edges of $K_{19}$.
List out all subsets of $V = \{1, \ldots, 19\}$ of size $R(4) = 18$.

\[ A_1, A_2, \ldots, A_{18}^{19}. \]

Since $|A_i| = R(4)$, each $A_i$ has a mono $K_4$. 

YOU"VE BEEN PUNKED It is quite possible that the mono $K_4$ from $A_3$ and the mono $K_4$ from $A_9$ are the same.

For the real proof, see next slide.
Proof of $K_{19}$ Two $K_4$ Theorem

**Thm** For all 2-cols of edges of $K_{19}$ there are 2 mono $K_4$'s

Assume you have a 2-col of the edges of $K_{19}$.
List out all subsets of $V = \{1, \ldots, 19\}$ of size $R(4) = 18$.

$$A_1, A_2, \ldots, A_{\binom{19}{18}}.$$ 

Since $|A_i| = R(4)$, each $A_i$ has a mono $K_4$.
So we get $\binom{19}{18}$ mono $K_4$'s. So we are almost there.
Thm For all 2-cols of edges of $K_{19}$ there are 2 mono $K_4$’s
Assume you have a 2-col of the edges of $K_{19}$.
List out all subsets of $V = \{1, \ldots, 19\}$ of size $R(4) = 18$.

$$A_1, A_2, \ldots, A^{19}_{18}.$$  

Since $|A_i| = R(4)$, each $A_i$ has a mono $K_4$.  
So we get $\binom{19}{18}$ mono $K_4$’s. So we are almost there.  
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Proof of $K_{19}$ Two $K_4$ Theorem

**Thm** For all 2-cols of edges of $K_{19}$ there are 2 mono $K_4$'s

Assume you have a 2-col of the edges of $K_{19}$.

List out all subsets of $V = \{1, \ldots, 19\}$ of size $R(4) = 18$.

$$A_1, A_2, \ldots, A_{\binom{19}{18}}.$$  

Since $|A_i| = R(4)$, each $A_i$ has a mono $K_4$.

So we get $\binom{19}{18}$ mono $K_4$'s. So we are almost there.

**YOU’VE BEEN PUNKED.** It is quite possible that the mono $K_4$ from $A_3$ and the mono $K_4$ from $A_9$ are the same.

For the real proof, see next slide.
Proof of $K_{19}$ Two $K_4$ Theorem (cont)

List out all subsets of $V = \{1, \ldots, 19\}$ of size $R(4) = 18$. 

1) Find a mono $K_4$ in $A_1$. Say its $\{16, 17, 18, 19\}$.
2) REMOVE all $A_i$'s that have all of $\{16, 17, 18, 19\}$.
There are $(19 - 4) - (15 - 4) = 15$ of these.
There are $(19 - 18) - (19 - 15) = 4$ left. Call them $B_1, B_2, B_3, B_4$.
3) Since $B_1$ has 18 vertices, there is a mono $K_4$ from $A_1$.
4) The mono $K_4$ from $A_1$, and the mono $K_4$ from $B$ are different.
Those are our 2 mono $K_4$'s.
Proof of $K_{19}$ Two $K_4$ Theorem (cont)

List out all subsets of $V = \{1, \ldots, 19\}$ of size $R(4) = 18$.

$$A_1, A_2, \ldots, A_{\binom{19}{18}}.$$
List out all subsets of $V = \{1, \ldots, 19\}$ of size $R(4) = 18$.

$A_1, A_2, \ldots, A_{\binom{19}{18}}$.

(There are just 19 of these, $A_i = \{1, \ldots, 19\} - \{i\}$.)
Proof of $K_{19}$ Two $K_4$ Theorem (cont)

List out all subsets of $V = \{1, \ldots, 19\}$ of size $R(4) = 18$.

$$A_1, A_2, \ldots, A_{19 \choose 18}.$$  

(There are just 19 of these, $A_i = \{1, \ldots, 19\} - \{i\}$.)

1) Find a mono $K_4$ in $A_1$. Say its $\{16, 17, 18, 19\}$.
Proof of $K_{19}$ Two $K_4$ Theorem (cont)

List out all subsets of $V = \{1, \ldots, 19\}$ of size $R(4) = 18$.

$$A_1, A_2, \ldots, A_{\binom{19}{18}}.$$  

(There are just 19 of these, $A_i = \{1, \ldots, 19\} - \{i\}$.)

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Proof of $K_{19}$ Two $K_4$ Theorem (cont)

List out all subsets of $V = \{1, \ldots, 19\}$ of size $R(4) = 18$.

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(There are just 19 of these, $A_i = \{1, \ldots, 19\} - \{i\}$.)

1) Find a mono $K_4$ in $A_1$. Say its $\{16, 17, 18, 19\}$.

2) REMOVE all $A_i$'s that have all of $\{16, 17, 18, 19\}$.

There are $19-4\choose 18-4 = \begin{pmatrix} 15 \\ 14 \end{pmatrix} = 15$ of these.
Proof of \( K_{19} \) Two \( K_4 \) Theorem (cont)

List out all subsets of \( V = \{1, \ldots, 19\} \) of size \( R(4) = 18 \).

\[ A_1, A_2, \ldots, A_{18}. \]

(There are just 19 of these, \( A_i = \{1, \ldots, 19\} - \{i\} \).)

1) Find a mono \( K_4 \) in \( A_1 \). Say its \( \{16, 17, 18, 19\} \).

2) REMOVE all \( A_i \)'s that have all of \( \{16, 17, 18, 19\} \).

There are \( \binom{19-4}{18-4} = \binom{15}{14} = 15 \) of these.

There are \( \binom{19}{18} - \binom{15}{14} = 19 - 15 = 4 \) left. Call then \( B_1, B_2, B_3, B_4 \).
Proof of $K_{19}$ Two $K_4$ Theorem (cont)

List out all subsets of $V = \{1, \ldots, 19\}$ of size $R(4) = 18$.

$$A_1, A_2, \ldots, A_{\binom{19}{18}}.$$ 

(There are just 19 of these, $A_i = \{1, \ldots, 19\} - \{i\}$.)

1) Find a mono $K_4$ in $A_1$. Say its $\{16, 17, 18, 19\}$.
2) REMOVE all $A_i$’s that have all of $\{16, 17, 18, 19\}$.
There are \((\binom{19}{18} - 4) = \binom{15}{14} = 15\) of these.
There are \((\binom{19}{18}) - (\binom{15}{14}) = 19 - 15 = 4\) left. Call then $B_1, B_2, B_3, B_4$.
3) Since $B_1$ has $18 = R(4)$ vertices, there is a mono $K_4$
Proof of $K_{19}$ Two $K_4$ Theorem (cont)

List out all subsets of $V = \{1, \ldots, 19\}$ of size $R(4) = 18$.

$$A_1, A_2, \ldots, A_{\binom{19}{18}}.$$ 

(There are just 19 of these, $A_i = \{1, \ldots, 19\} - \{i\}$.)

1) Find a mono $K_4$ in $A_1$. Say its $\{16, 17, 18, 19\}$.

2) REMOVE all $A_i$’s that have all of $\{16, 17, 18, 19\}$.
There are $\binom{19-4}{18-4} = \binom{15}{14} = 15$ of these.
There are $\binom{19}{18} - \binom{15}{14} = 19 - 15 = 4$ left. Call then $B_1, B_2, B_3, B_4$.

3) Since $B_1$ has $18 = R(4)$ vertices, there is a mono $K_4$.

4) The mono $K_4$ from $A_1$, and the mono $K_4$ from $B$ are different.
Proof of $K_{19}$ Two $K_4$ Theorem (cont)

List out all subsets of $V = \{1, \ldots, 19\}$ of size $R(4) = 18$.

$$A_1, A_2, \ldots, A_{\binom{19}{18}}.$$ 

(There are just 19 of these, $A_i = \{1, \ldots, 19\} - \{i\}$.)

1) Find a mono $K_4$ in $A_1$. Say its $\{16, 17, 18, 19\}$.

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There are $\binom{19}{18} - \binom{15}{14} = 19 - 15 = 4$ left. Call then $B_1, B_2, B_3, B_4$.

3) Since $B_1$ has 18 = $R(4)$ vertices, there is a mono $K_4$

4) The mono $K_4$ from $A_1$, and the mono $K_4$ from $B$ are different.

Those are our 2 mono $K_4$’s.
Can the Proof Give 3 mono $K_4$’s?

We show that the technique to get 2 mono $K_4$’s cannot be extended to give 3 mono $K_4$’s.
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So we have removed all $A \subseteq \{1, \ldots, 19\}$ of size 18 where $A$ has all of $\{16, 17, 18, 19\}$, or $A$ has all of $\{12, 13, 14, 15\}$.
Whats left?
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So we have removed all $A \subseteq \{1, \ldots, 19\}$ of size 18 where

$A$ has all of $\{16, 17, 18, 19\}$, or

$A$ has all of $\{12, 13, 14, 15\}$.

What's left? All $A \subseteq \{1, \ldots, 19\}$ of size 18 that are missing at least one of $\{16, 17, 18, 19\}$ and at least one of $\{12, 13, 14, 15\}$. 

If $A \subseteq \{1, \ldots, 19\}$ is missing two elements it is of size 17. Hence there are none left.

Note We only showed that the proof cannot be extended. As noted above any 2-col of $K_{18}$ has 9 mono $K_4$’s.
Can the Proof Give 3 mono $K_4$’s?

We show that the technique to get 2 mono $K_4$’s cannot be extended to give 3 mono $K_4$’s.

Assume that the first $K_4$ is $\{16, 17, 18, 19\}$

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We show that the technique to get 2 mono $K_4$’s cannot be extended to give 3 mono $K_4$’s.

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Assume that the second $K_4$ is $\{12, 13, 14, 15\}$

So we have removed all $A \subseteq \{1, \ldots, 19\}$ of size 18 where $A$ has all of $\{16, 17, 18, 19\}$, or $A$ has all of $\{12, 13, 14, 15\}$.

What’s left? All $A \subseteq \{1, \ldots, 19\}$ of size 18 that are missing at least one of $\{16, 17, 18, 19\}$ and at least one of $\{12, 13, 14, 15\}$. If $A \subseteq \{1, \ldots, 19\}$ is missing two elements it is of size 17. Hence there are none left.
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We show that the technique to get 2 mono $K_4$’s cannot be extended to give 3 mono $K_4$’s.

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So we have removed all $A \subseteq \{1, \ldots, 19\}$ of size 18 where $A$ has all of $\{16, 17, 18, 19\}$, or $A$ has all of $\{12, 13, 14, 15\}$.

What’s left? All $A \subseteq \{1, \ldots, 19\}$ of size 18 that are missing at least one of $\{16, 17, 18, 19\}$ and at least one of $\{12, 13, 14, 15\}$.

If $A \subseteq \{1, \ldots, 19\}$ is missing two elements it is of size 17. Hence there are none left.

**Note** We only showed that the proof cannot be extended. As noted above any 2-col of $K_{18}$ has 9 mono $K_4$’s.
Want $n$ such that $\forall$ 2-col $\exists$ 3 Mono $K_4$'s
Want $n$ such that $\forall$ 2-col $\exists 3$ Mono $K_4$’s

$\forall$ 2-col of $K_n$ $\exists$ 3 mono $K_4$’s.
Want $n$ such that $\forall$ 2-col $\exists$ 3 Mono $K_4$’s

$\forall$ 2-col of $K_n$ $\exists$ 3 mono $K_4$’s.

List out all subsets of $V = \{1, \ldots, n\}$ of size $R(4) = 18$. 
Want \( n \) such that \( \forall \) 2-col \( \exists 3 \) Mono \( K_4 \)'s

\( \forall \) 2-col of \( K_n \) \( \exists 3 \) mono \( K_4 \)'s.

List out all subsets of \( V = \{1, \ldots, n\} \) of size \( R(4) = 18 \).

\( A_1, A_2, \ldots, A_{\binom{n}{18}} \).
Want \( n \) such that \( \forall \) 2-col \( \exists \) 3 Mono \( K_4 \)'s.

\[ \forall \text{2-col of } K_n \ \exists \text{3 mono } K_4 \text{'s.} \]

List out all subsets of \( V = \{1, \ldots, n\} \) of size \( R(4) = 18 \).

\[ A_1, A_2, \ldots, A_{\binom{n}{18}}. \]

1) Find a mono \( K_4 \) in \( A_1 \). Say its \( \{x_1, x_2, x_3, x_4\} \).
Want \( n \) such that \( \forall \text{ 2-col } \exists 3 \text{ Mono } K_4 \text{'s} \)

\( \forall \text{ 2-col of } K_n \exists 3 \text{ mono } K_4 \text{'s}. \)

List out all subsets of \( V = \{1, \ldots, n\} \) of size \( R(4) = 18 \).

\[ A_1, A_2, \ldots, A_{\binom{n}{18}}. \]

1) Find a mono \( K_4 \) in \( A_1 \). Say its \( \{x_1, x_2, x_3, x_4\} \).
2) REMOVE all \( A_i \)'s that have all of \( \{x_1, x_2, x_3, x_4\} \).

\( (n-4)_{18} = (n-4)_{14} \) of these. There are \( (n)_{18} - 2(n-4)_{14} \) left.

3) Find a mono \( K_4 \) in one of the sets left. Now have 3. But . . .
Want \( n \) such that \( \forall \text{2-col} \ \exists 3 \text{ Mono} \ K_4 \text{'s} \)

\[ \forall \text{2-col of } K_n \ \exists 3 \text{ mono } K_4 \text{'s}. \]

List out all subsets of \( V = \{1, \ldots, n\} \) of size \( R(4) = 18 \).

\[ A_1, A_2, \ldots, A_{\binom{n}{18}}. \]

1) Find a mono \( K_4 \) in \( A_1 \). Say its \( \{x_1, x_2, x_3, x_4\} \).

2) REMOVE all \( A_i \)'s that have all of \( \{x_1, x_2, x_3, x_4\} \).

\[ \binom{n-4}{18-4} = \binom{n-4}{14} \] of these. There are \( \binom{n}{18} - \binom{n-4}{14} \) left.
Want \( n \) such that \( \forall \) 2-col \( \exists 3 \) Mono \( K_4 \)'s

\[ \forall \text{ 2-col of } K_n \exists 3 \text{ mono } K_4 \text{'s.} \]

List out all subsets of \( V = \{1, \ldots, n\} \) of size \( R(4) = 18 \).

\[ A_1, A_2, \ldots, A_{\binom{n}{18}}. \]

1) Find a mono \( K_4 \) in \( A_1 \). Say its \( \{x_1, x_2, x_3, x_4\} \).

2) REMOVE all \( A_i \)'s that have all of \( \{x_1, x_2, x_3, x_4\} \).

\[ \binom{n-4}{18-4} = \binom{n-4}{14} \] of these. There are \( \binom{n}{18} - \binom{n-4}{14} \) left.

3) Find a mono \( K_4 \) in one of the sets left. Say its \( \{y_1, y_2, y_3, y_4\} \).
Want $n$ such that $\forall$ 2-col $\exists$ 3 Mono $K_4$’s

$\forall$ 2-col of $K_n$ $\exists$ 3 mono $K_4$’s.

List out all subsets of $V = \{1, \ldots, n\}$ of size $R(4) = 18$.

$A_1, A_2, \ldots, A_{\binom{n}{18}}$.

1) Find a mono $K_4$ in $A_1$. Say its $\{x_1, x_2, x_3, x_4\}$.
2) REMOVE all $A_i$’s that have all of $\{x_1, x_2, x_3, x_4\}$. $\binom{n-4}{18-4} = \binom{n-4}{14}$ of these. There are $\binom{n}{18} - \binom{n-4}{14}$ left.
3) Find a mono $K_4$ in one of the sets left. Say its $\{y_1, y_2, y_3, y_4\}$.
4) REMOVE all $A_i$’s that have all of $\{y_1, y_2, y_3, y_4\}$. 

. . .
Want \( n \) such that \( \forall \) 2-col \( \exists \) 3 Mono \( K_4 \)'s

\[ \forall \text{ 2-col of } K_n \ \exists \ 3 \text{ mono } K_4 \text{'s}. \]

List out all subsets of \( V = \{1, \ldots, n\} \) of size \( R(4) = 18 \).

\[ A_1, A_2, \ldots, A_{n \choose 18}. \]

1) Find a mono \( K_4 \) in \( A_1 \). Say its \( \{x_1, x_2, x_3, x_4\} \).

2) REMOVE all \( A_i \)'s that have all of \( \{x_1, x_2, x_3, x_4\} \).
\[ \left( \begin{array}{c} n-4 \\ 18-4 \end{array} \right) = \left( \begin{array}{c} n-4 \\ 14 \end{array} \right) \] of these. There are \( \left( \begin{array}{c} n \\ 18 \end{array} \right) - \left( \begin{array}{c} n-4 \\ 14 \end{array} \right) \) left.

3) Find a mono \( K_4 \) in one of the sets left. Say its \( \{y_1, y_2, y_3, y_4\} \).

4) REMOVE all \( A_i \)'s that have all of \( \{y_1, y_2, y_3, y_4\} \).
\[ \left( \begin{array}{c} n-4 \\ 18-4 \end{array} \right) = \left( \begin{array}{c} n-4 \\ 14 \end{array} \right) \] of these. There are \( \left( \begin{array}{c} n \\ 18 \end{array} \right) - 2 \left( \begin{array}{c} n-4 \\ 14 \end{array} \right) \) left.
Want $n$ such that $\forall$ 2-col $\exists$ 3 Mono $K_4$'s

$\forall$ 2-col of $K_n \exists$ 3 mono $K_4$'s.

List out all subsets of $V = \{1, \ldots, n\}$ of size $R(4) = 18$.

$A_1, A_2, \ldots, A_{\binom{n}{18}}$.

1) Find a mono $K_4$ in $A_1$. Say its $\{x_1, x_2, x_3, x_4\}$.

2) REMOVE all $A_i$'s that have all of $\{x_1, x_2, x_3, x_4\}$.

$\binom{n-4}{18-4} = \binom{n-4}{14}$ of these. There are $\binom{n}{18} - \binom{n-4}{14}$ left.

3) Find a mono $K_4$ in one of the sets left. Say its $\{y_1, y_2, y_3, y_4\}$.

4) REMOVE all $A_i$'s that have all of $\{y_1, y_2, y_3, y_4\}$.

$\binom{n-4}{18-4} = \binom{n-4}{14}$ of these. There are $\binom{n}{18} - 2\binom{n-4}{14}$ left.

5) Find a mono $K_4$ in one of the sets left. Now have 3. But....
Want 3 Mono $K_4$'s (cont)

Need
Want 3 Mono $K_4$’s (cont)

Need

$$\binom{n}{18} - 2\binom{n-4}{14} \geq 1$$
Want 3 Mono $K_4$’s (cont)

Need

\[
\binom{n}{18} - 2\binom{n-4}{14} \geq 1
\]

\[
\binom{n}{18} - 2\binom{n-4}{14} > 0
\]
Want 3 Mono $K_4$'s (cont)

Need

\[
\binom{n}{18} - 2 \binom{n - 4}{14} \geq 1
\]

\[
\binom{n}{18} - 2 \binom{n - 4}{14} > 0
\]

\[
\frac{n!}{18!(n - 18)!} > 2 \frac{(n - 4)!}{14!(n - 18)!}
\]
Want 3 Mono $K_4$’s (cont)

Need

\[
\binom{n}{18} - 2 \binom{n-4}{14} \geq 1
\]

\[
\binom{n}{18} - 2 \binom{n-4}{14} > 0
\]

\[
\frac{n!}{18!(n-18)!} > 2 \frac{(n-4)!}{14!(n-18)!}
\]

\[
\frac{n!}{18 \times 17 \times 16 \times 15} > 2(n-4)!
\]
Want 3 Mono $K_4$’s (cont)

Need

$$\binom{n}{18} - 2\binom{n-4}{14} \geq 1$$

$$\binom{n}{18} - 2\binom{n-4}{14} > 0$$

$$\frac{n!}{18!(n-18)!} > 2 \frac{(n-4)!}{14!(n-18)!}$$

$$\frac{n!}{18 \times 17 \times 16 \times 15} > 2(n-4)!$$

$$n(n-1)(n-2)(n-3) > 2 \times 18 \times 17 \times 16 \times 15$$
Want 3 Mono $K_4$’s (cont)

\[n(n - 1)(n - 2)(n - 3) > 2 \times 18 \times 17 \times 16 \times 15 = 146889.\]
Want 3 Mono $K_4$’s (cont)

\[ n(n-1)(n-2)(n-3) > 2 \times 18 \times 17 \times 16 \times 15 = 146889. \]

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<td>20</td>
<td>116280</td>
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<td>21</td>
<td>143640</td>
</tr>
<tr>
<td>22</td>
<td>175560</td>
</tr>
</tbody>
</table>
Want 3 Mono $K_4$’s (cont)

\[ n(n-1)(n-2)(n-3) > 2 \times 18 \times 17 \times 16 \times 15 = 146889. \]

<table>
<thead>
<tr>
<th>$n$</th>
<th>$n(n-1)(n-2)(n-3)$</th>
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<tbody>
<tr>
<td>19</td>
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</tbody>
</table>

**Thm** \( \forall \) 2-cols of the edges of $K_{22}$ \( \exists \) 3 mono $K_4$’s.
Want $m$ Mono $K_4$'s

The key to the prior proof is how many $A_i$'s do you remove.
Want $m$ Mono $K_4$’s

The key to the prior proof is how many $A_i$’s do you remove. We removed \( \binom{n-4}{18-4} = \binom{n-4}{14} \) in each iteration.
Want $m$ Mono $K_4$’s

The key to the prior proof is how many $A_i$’s do you remove. We removed \((\binom{n-4}{18-4}) = \binom{n-4}{14}\) in each iteration.

**Thm** Let \(m, n \geq \mathbb{N}\). Assume \(\binom{n}{18} - m\binom{n-4}{14} \geq 1\). For any 2-col of $K_n$ there exists $m + 1$ mono $K_4$’s.
Want $m$ Mono $K_4$’s

The key to the prior proof is how many $A_i$’s do you remove. We removed $\binom{n-4}{18-4} = \binom{n-4}{14}$ in each iteration.

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Subsets of $V = \{1, \ldots, n\}$, size $R(4) = 18$: $A_1, A_2, \ldots, A_{\binom{n}{18}}$. 
Want $m$ Mono $K_4$'s

The key to the prior proof is how many $A_i$'s do you remove. We removed \( \binom{n-4}{18-4} = \binom{n-4}{14} \) in each iteration.

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Subsets of $V = \{1, \ldots, n\}$, size $R(4) = 18$: $A_1, A_2, \ldots, A_{\binom{n}{18}}$.

1) SETA = \{ $A_1, A_2, \ldots, A_{\binom{n}{18}}$ \}. SETK4 = \emptyset.
Want $m$ Mono $K_4$’s

The key to the prior proof is how many $A_i$’s do you remove. We removed \( \binom{n-4}{18-4} = \binom{n-4}{14} \) in each iteration.

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Subsets of $V = \{1, \ldots, n\}$, size $R(4) = 18$: $A_1, A_2, \ldots, A_\binom{n}{18}$.

1) $\text{SETA} = \{A_1, A_2, \ldots, A_\binom{n}{18}\}$. $\text{SETK4} = \emptyset$.

2) Take arb $A \in \text{SETA}$. $\exists$ mono $K_4$ in $A$, $K = \{x_1, x_2, x_3, x_4\}$. 
Want \( m \) Mono \( K_4 \)'s

The key to the prior proof is how many \( A_i \)'s do you remove. We removed \( \binom{n-4}{18-4} = \binom{n-4}{14} \) in each iteration.

**Thm** Let \( m, n \geq \mathbb{N} \). Assume \( \binom{n}{18} - m \binom{n-4}{14} \geq 1 \). For any 2-col of \( K_n \) there exists \( m + 1 \) mono \( K_4 \)'s.

Subsets of \( V = \{1, \ldots, n\} \), size \( R(4) = 18 \): \( A_1, A_2, \ldots, A_{\binom{n}{18}} \).

1) \( \text{SETA} = \{A_1, A_2, \ldots, A_{\binom{n}{18}}\} \). \( \text{SETK4} = \emptyset \).

2) Take arb \( A \in \text{SETA} \). \( \exists \) mono \( K_4 \) in \( A \), \( K = \{x_1, x_2, x_3, x_4\} \).

\[ \text{SETK4} = \text{SETK4} \cup \{K\} \]
Want $m$ Mono $K_4$'s

The key to the prior proof is how many $A_i$’s do you remove. We removed $\binom{n-4}{18-4} = \binom{n-4}{14}$ in each iteration.

**Thm** Let $m, n \geq \mathbb{N}$. Assume $\binom{n}{18} - m\binom{n-4}{14} \geq 1$. For any 2-col of $K_n$ there exists $m + 1$ mono $K_4$’s.

Subsets of $V = \{1, \ldots, n\}$, size $R(4) = 18$: $A_1, A_2, \ldots, A_{\binom{n}{18}}$.

1) $\text{SETA} = \{A_1, A_2, \ldots, A_{\binom{n}{18}}\}$. $\text{SETK4} = \emptyset$.

2) Take arb $A \in \text{SETA}$. $\exists$ mono $K_4$ in $A$, $K = \{x_1, x_2, x_3, x_4\}$.
   - $\text{SETK4} = \text{SETK4} \cup \{K\}$.
   - $\text{SETA} = \text{SETA} - \{A \in \text{SETA} : x_1, x_2, x_3, x_4 \in A\}$. 

Since $\binom{n}{18} - m\binom{n-4}{14} \geq 1$ this process can go for $\geq m + 1$ iterations and produce $\geq m + 1$ mono $K_4$’s.
Want $m$ Mono $K_4$’s

The key to the prior proof is how many $A_i$’s do you remove. We removed $\binom{n-4}{18-4} = \binom{n-4}{14}$ in each iteration.

**Thm** Let $m, n \geq \mathbb{N}$. Assume $\binom{n}{18} - m \binom{n-4}{14} \geq 1$. For any 2-col of $K_n$ there exists $m + 1$ mono $K_4$’s.

Subsets of $V = \{1, \ldots, n\}$, size $R(4) = 18$: $A_1, A_2, \ldots, A_{\binom{n}{18}}$.

1) SETA = $\{A_1, A_2, \ldots, A_{\binom{n}{18}}\}$. SETK4 = $\emptyset$.

2) Take arb $A \in$ SETA. $\exists$ mono $K_4$ in $A$, $K = \{x_1, x_2, x_3, x_4\}$.
   
   ▶ SETK4 = SETK4 $\cup \{K\}$.
   
   ▶ SETA = SETA $\setminus \{A \in$ SETA : $x_1, x_2, x_3, x_4 \in A\}$.

3) If SETA $\neq \emptyset$ then go to step 2. Else STOP.
Want $m$ Mono $K_4$'s

The key to the prior proof is how many $A_i$’s do you remove. We removed \( \binom{n-4}{18-4} = \binom{n-4}{14} \) in each iteration.

**Thm** Let $m, n \geq \mathbb{N}$. Assume \( \binom{n}{18} - m\binom{n-4}{14} \geq 1 \). For any 2-col of $K_n$ there exists $m + 1$ mono $K_4$’s.

Subsets of $V = \{1, \ldots, n\}$, size $R(4) = 18$: $A_1, A_2, \ldots, A_{\binom{n}{18}}$.

1) $\text{SETA} = \{A_1, A_2, \ldots, A_{\binom{n}{18}}\}$. $\text{SETK4} = \emptyset$.

2) Take arb $A \in \text{SETA}$. $\exists$ mono $K_4$ in $A$, $K = \{x_1, x_2, x_3, x_4\}$.
   - $\text{SETK4} = \text{SETK4} \cup \{K\}$.
   - $\text{SETA} = \text{SETA} - \{A \in \text{SETA} : x_1, x_2, x_3, x_4 \in A\}$.

3) If $\text{SETA} \neq \emptyset$ then go to step 2. Else STOP.

Since \( \binom{n}{18} - m\binom{n-4}{14} \geq 1 \) this process can go for $\geq m + 1$ iterations and produce $\geq m + 1$ mono $K_4$’s.
We just proved that for all \( n, m \in \mathbb{N} \):

**Thm** If \( \binom{n}{18} - m \binom{n-4}{14} \geq 1 \) then \( \forall \) 2-col of \( K_n \) \( \exists \) \( m + 1 \) mono \( K_4 \)’s.
We just proved that for all $n, m \in \mathbb{N}$:

**Thm** If $\binom{n}{18} - m\binom{n-4}{14} \geq 1$ then $\forall$ 2-col of $K_n \exists m + 1$ mono $K_4$'s.

We want $m$ as a function of $n$. 

We state a theorem which expresses this in several ways, on the next slide.
Want $m$ Mono $K_4$’s (cont)

We just proved that for all $n, m \in \mathbb{N}$:

**Thm** If $\binom{n}{18} - m\binom{n-4}{14} \geq 1$ then $\forall$ 2-col of $K_n \exists m + 1$ mono $K_4$’s.

We want $m$ as a function of $n$.

$$\binom{n}{18} - m\binom{n-4}{14} \geq 0$$
Want $m$ Mono $K_4$’s (cont)

We just proved that for all $n, m \in \mathbb{N}$:

**Thm** If $\binom{n}{18} - m\binom{n-4}{14} \geq 1$ then $\forall$ 2-col of $K_n \exists m + 1$ mono $K_4$’s.

We want $m$ as a function of $n$.

$$\binom{n}{18} - m\binom{n-4}{14} \geq 0$$

$$m \leq \frac{\binom{n}{18}}{\binom{n-4}{14}} = \frac{n!}{18!(n-18)!} \frac{14!(n-18)!}{(n-4)!} = \frac{n(n-1)(n-2)(n-3)}{18 \times 17 \times 16 \times 15}$$
We just proved that for all $n, m \in \mathbb{N}$:

**Thm** If $\binom{n}{18} - m\binom{n-4}{14} \geq 1$ then $\forall$ 2-col of $K_n \exists m + 1$ mono $K_4$’s.

We want $m$ as a function of $n$.

$$\binom{n}{18} - m\binom{n-4}{14} \geq 0$$

$$m \leq \frac{\binom{n}{18}}{\binom{n-4}{14}} = \frac{n!}{18!(n-18)!} \frac{14!(n-18)!}{(n-4)!} = \frac{n(n-1)(n-2)(n-3)}{18 \times 17 \times 16 \times 15}$$

We state a theorem which expresses this in several ways, on the next slide.
More Versions

\[ \text{Let } n \geq N. \]
\[ \forall 2\text{-col of } K_n \text{ the following happens.} \]
1) There are \( \lfloor \frac{n}{18} \rfloor \left( \frac{n}{18} - 4 \right) \left( \frac{n}{18} - 14 \right) + 1 \) mono \( K_4 \)'s.
2) There are \( n \left( n - 1 \right) \left( n - 2 \right) \left( n - 3 \right) \times 18 \times 17 \times 16 \times 15 + 1 \) mono \( K_4 \)'s.
3) There are \( n \left( n - 1 \right) \left( n - 2 \right) \left( n - 3 \right) 73440 + 1 \) mono \( K_4 \)'s.
4) There are \( n^4 73440 - n^3 12240 + \Omega(n^2) \) mono \( K_4 \)'s.
Thm Let $n \geq \mathbb{N}$. ∀ 2-col of $K_n$ the following happens.
Thm Let $n \geq \mathbb{N}$. ∀ 2-col of $K_n$ the following happens.

1) There are $\left\lfloor \frac{n}{18} \cdot \frac{n}{14} \right\rfloor + 1$ mono $K_4$’s.
**Thm** Let \( n \geq \mathbb{N} \). \( \forall \) 2-col of \( K_n \) the following happens.

1) There are \( \left\lfloor \frac{n}{18} \cdot \frac{n}{14} \right\rfloor + 1 \) mono \( K_4 \)'s.

2) There are \( \frac{n(n-1)(n-2)(n-3)}{18 \times 17 \times 16 \times 15} + 1 \) mono \( K_4 \)'s.
Thm Let $n \geq \mathbb{N}$. \forall 2\text{-}col of $K_n$ the following happens.

1) There are $\left\lfloor \frac{n}{18} \right\rfloor \frac{n}{n-4} + 1$ mono $K_4$'s.

2) There are $\frac{n(n-1)(n-2)(n-3)}{18 \times 17 \times 16 \times 15} + 1$ mono $K_4$'s.

3) There are $\frac{n(n-1)(n-2)(n-3)}{73440} + 1$ mono $K_4$'s.
**Thm** Let $n \geq \mathbb{N}$. ∀ 2-col of $K_n$ the following happens.

1) There are $\lfloor \frac{n}{18} \rfloor \times \frac{n}{14} + 1$ mono $K_4$'s.

2) There are $\frac{n(n-1)(n-2)(n-3)}{18 \times 17 \times 16 \times 15} + 1$ mono $K_4$'s.

3) There are $\frac{n(n-1)(n-2)(n-3)}{73440} + 1$ mono $K_4$'s.

4) There are $\frac{n^4}{73440} - \frac{n^3}{12240} + \Omega(n^2)$ mono $K_4$'s.
Another Way to Phrase The Results

**Thm** \( \forall 2\text{-cols of } K_n \; \exists \sim \frac{n^3}{24} \text{ mono } K_3. \)
Another Way to Phrase The Results

Thm ∀ 2-cols of $K_n \exists \sim \frac{n^3}{24}$ mono $K_3$.
In $K_n$ there are $\binom{n}{3}$ triples.
Another Way to Phrase The Results

\textbf{Thm} \ \forall \ 2\text{-cols of } K_n \ \exists \ \sim \ \frac{n^3}{24} \text{ mono } K_3.

In $K_n$ there are $\binom{n}{3}$ triples.

We want to know the \textbf{fraction} of them that are mono.
Another Way to Phrase The Results

**Thm** $\forall$ 2-cols of $K_n \exists \sim \frac{n^3}{24}$ mono $K_3$.

In $K_n$ there are $(\binom{n}{3})$ triples.
We want to know the fraction of them that are mono.

**Thm** $\forall$ 2-cols of $K_n \exists \sim \frac{1}{8} \binom{n}{3}$ mono $K_3$. 

There are $\sim n^4 \frac{73440}{79}$ mono $K_4$'s.
We rephrase this as what fraction of the $(\binom{n}{4})$ $K_4$'s are mono.

There are $\frac{1}{3060} \binom{n}{4}$ mono $K_4$'s.
Another Way to Phrase The Results

**Thm** $\forall$ 2-cols of $K_n \ni \sim \frac{n^3}{24}$ mono $K_3$.

In $K_n$ there are $\binom{n}{3}$ triples.

We want to know the fraction of them that are mono.

**Thm** $\forall$ 2-cols of $K_n \ni \sim \frac{1}{8} \binom{n}{3}$ mono $K_3$.

There are $\sim \frac{n^4}{73440}$ mono $K_4$'s.
Another Way to Phrase The Results

**Thm** \( \forall \text{ 2-cols of } K_n \exists \sim \frac{n^3}{24} \text{ mono } K_3. \)

In \( K_n \) there are \( \binom{n}{3} \) triples.

We want to know the *fraction* of them that are mono.

**Thm** \( \forall \text{ 2-cols of } K_n \exists \sim \frac{1}{8} \binom{n}{3} \text{ mono } K_3. \)

There are \( \sim \frac{n^4}{73440} \) mono \( K_4 \)'s.

We rephrase this as what fraction of the \( \binom{n}{4} \) \( K_4 \)'s are mono.
Another Way to Phrase The Results

Thm $\forall$ 2-cols of $K_n \exists \sim \frac{n^3}{24}$ mono $K_3$.
In $K_n$ there are $\binom{n}{3}$ triples.
We want to know the fraction of them that are mono.

Thm $\forall$ 2-cols of $K_n \exists \sim \frac{1}{8}\binom{n}{3}$ mono $K_3$.
There are $\sim \frac{n^4}{73440}$ mono $K_4$'s.
We rephrase this as what fraction of the $\binom{n}{4}$ $K_4$'s are mono.
There are $\frac{1}{3060}\binom{n}{4}$ mono $K_4$'s.
Generalize

Left to the reader
Generalize

Left to the reader

1. Generalize to mono $K_m$. 
Generalize

Left to the reader

1. Generalize to mono $K_m$.
2. Generalize to $c$ colors.
Generalize

Left to the reader

1. Generalize to mono $K_m$.
2. Generalize to $c$ colors.
3. Generalize to $c$ colors and mono $K_m$. 