One Triangle, Two Triangles

William Gasarch

Lets Party Like Its 2019

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Theorem For all 2-coloring of the edges of K_6 there is a mono K_3.

Trivial Theorem, Non Trivial Extension

Theorem For all 2-cols of edges of K_{12} there are 2 mono K_3's
Question Find n such that

1. For all 2-coloring of the edges of K_n there are 2 mono K_3's
2. There exists a 2-coloring of the edges of K_{n-1} that does not have 2 mono K_3's.

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1. For all 2-coloring of the edges of K_6 there are **2** mono K_3's
2. There exists a 2-coloring of the edges of K_5 that does not have any mono K_3's.

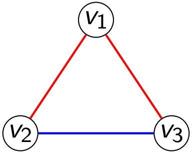
Proof of K_6 Two Triangles Theorem

Theorem For all 2-cols of edges of K_6 there are 2 mono K_3's
Proof Let COL be a 2-coloring of the edges of K_6.
Let R, B, M, be the SET of **RED**, **BLUE**, and **MIXED** triangles.

$$|R| + |B| + |M| = \binom{6}{3} = 20.$$

We show that $|M| \leq 18$, so $|R| + |B| \geq 2$.

A Mixed Triangle Has a Vertex Such That



- (v_2, v_1) is red, (v_2, v_3) is blue. View this as $(v_2, \{v_1, v_3\})$.
- (v_3, v_1) is red, (v_3, v_2) is blue. View this as $(v_3, \{v_1, v_2\})$.

Map ZAN to M

Def A **Zan** is an element $(v, \{u, w\}) \in V \times \binom{V}{2}$ such that $v \notin \{u, w\}$ and $COL(v, u) \neq COL(v, w)$. ZAN is the set of Zan's.

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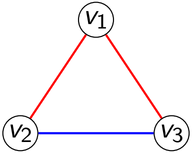

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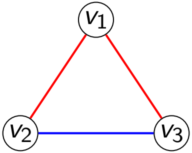

$(v_2, \{v_1, v_3\})$ and $(v_3, \{v_1, v_2\})$.

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So v contributes $\deg_R(v) \times \deg_B(v)$.

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So there are at least 2 Mono Triangles.

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$$|M| = |ZAN|/2 \le \frac{(n-1)^2 n}{8}$$

Finishing Up The Proof

Recap

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$$|R| + |B| + |M| = \binom{n}{3} = \frac{n(n-1)(n-2)}{6} \text{ hence}$$

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$$= \frac{n^3}{24} - \frac{n^2}{4} + \frac{5n}{24}$$

Can This Be Improved?

The bound is known to be tight.

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   This is interesting to know but the proof is not mathematically interesting.
2. I will present a mathematically interesting proof of the following:
   For all 2-colorings of K_{19} there are TWO mono K_4's.

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So we get $\binom{19}{18}$ mono K_4's. So we are almost there.

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For the real proof, see next slide.

Proof of K_{19} Two K_4 Theorem (cont)

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(There are just 19 of these, $A_i = \{1, \ldots, 19\} - \{i\}$.)

Proof of K_{19} Two K_4 Theorem (cont)

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4) The mono K_4 from A_1, and the mono K_4 from B are different. Those are our 2 mono K_4's.

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Note We only showed that the **proof** cannot be extended. As noted above any 2-coloring of K_{18} has 9 mono K_4's.

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Need $\binom{n}{18} - 2\binom{n-4}{14} \geq 1$. Next slide.

Want Three Mono K_4's (cont)

$$\binom{n}{18} - 2\binom{n-4}{14} \geq 1$$

$$\binom{n}{18} - 2\binom{n-4}{14} > 0$$

$$\frac{n!}{18!(n-18)!} > 2\frac{(n-4)!}{14!(n-18)!}$$

$$\frac{n!}{18 \times 17 \times 16 \times 15} > 2(n-4)!$$

$$n(n-1)(n-2)(n-3) > 2 \times 18 \times 17 \times 16 \times 15$$

Want Three Mono K_4's (cont)

$$n(n-1)(n-2)(n-3) > 2 \times 18 \times 17 \times 16 \times 15 = 146889.$$

n	$n(n-)1(n-2)(n-3)$
19	93024
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Since $\binom{n}{18} - m\binom{n-4}{14} \geq 0$ this process can go for $\geq m$ iterations and produce $\geq m$ mono K_4's.

Want m Mono K_4's (cont)

We just proved:
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We want m as a function of n.

$$\binom{n}{18} - m\binom{n-4}{14} \geq 0$$

$$m \leq \frac{\binom{n}{18}}{\binom{n-4}{14}}$$

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Thm Let $n \geq \mathbb{N}$. $\forall$ 2-coloring of K_n $\exists$ $\left\lfloor \frac{\binom{n}{18}}{\binom{n-4}{14}} \right\rfloor$ mono K_4's.