**The Convex Polygon Problem**

# 1 Introduction

These notes are helpful if you both watched the recording and attended class (by zoom). Otherwise I doubt they are helpful.

**Convention 1.1** Every time we mention a set of points in $\mathbb{R}^2$ they have no three colinear

# 2 Happy Ending Theorem

**Def 2.1** Let $A \subseteq \mathbb{R}^2$ of size $k$. The points in $A$ form a *convex $k$-gon* if for every $x, y, z \in A$, there is no point of $A$ in the triangle formed by $x, y, z$. Henceforth we just say *$k$-gon*.

**Theorem 2.2** *(Esther Klein) For every 5 points in $\mathbb{R}^2$ there exists a 4-gon.*

**Theorem 2.3** *(Erdös and Szekeres) For all $k \ge 3$ there exists $n$ such that for every set of $n$ points in $\mathbb{R}^2$ there exists $k$ of them that form a $k$-gon.*

**Sketch:**

$k = 3$: Take $n = 3$.

$k = 4$: Take $n = 5$ and use Klein's Theorem.

We assume $k \ge 5$.

We went over three proofs that used the following three colorings.

The points are $p_1, \ldots, p_n$. The ordering on the points is arbitrary; however, for the third proof we need the ordering.

**Proof 1:** $n = R_4(k)$. We have any $n$ points in $\mathbb{R}^2$

$COL(w, x, y, z)$ is RED if the for points form a 4-gon, and BLUE if they do not.

The homog set can't be BLUE since if was then there would be $k \ge 5$ points such that NO 4-subset was a 4-gon, which contradicts Klein's Theorem.

Hence there are $k$ points so that every set of 4 of them forms a 4-gon. One can show that the entire set is a $k$-gon.

**Proof 1':** We can use $n = R_4(k, 5)$ which is the smallest $n$ such that any 2-coloring of $\binom{[n]}{4}$ has either a RED Homog set of size $k$ or a BLUE homog set of size 5.

**Proof 2:** $n = R_3(k)$. We have any $n$ points in $\mathbb{R}^2$

$COL(w, x, y)$ is RED if their is an EVEN number of points inside the $x, y, z$ triangle, BLUE otherwise.

Both cases are possible. One can show that in either case the set is a $k$-gon using a parity argument.

**Proof 3:** $n = R_3(k)$. We have any $n$ points in $\mathbb{R}^2$

$COL(p_i, p_j, p_k)$ where $i < j < k$ is RED if $p_i, p_j, p_k$ is clockwise, and BLUE if counterclockwise.

Some cases, finishing the proof will be on a HW.

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These bounds are quite large. The following upper and lower bounds are known.

**Theorem 2.4**

1. *(Erdös and Szekeres) For all $k \geq 3$ there exists $n \leq \binom{2n-4}{n-2} + 1 = 4^{n+o(n)}$ such that for every set of $n$ points in $\mathbb{R}^2$ there exists $k$ of them that form a $k$-gon.*

2. *(Andrew Suk) For all $k \geq 3$ there exists $n \leq 2^{n+o(n)}$ such that for every set of $n$ points in $\mathbb{R}^2$ there exists $k$ of them that form a $k$-gon.*

3. (a) *For all sets of 3 points in $\mathbb{R}^2$ there exists a subset of 3 that form a 3-gon (this is trivial). This is tight.*

   (b) *For all sets of 5 points in $\mathbb{R}^2$ there exists a subset of 4 that form a 4-gon. This is tight.*

   (c) *For all sets of 9 points in $\mathbb{R}^2$ there exists a subset of 5 that form a 5-gon. This is tight.*

   (d) *For all sets of 17 points in $\mathbb{R}^2$ there exists a subset of 6 that form a 6-gon. This is tight.*

4. *For all $k \geq 3$ there exists a set of $2^{k-2}$ points such that there is NO subset of size $k$ that form a $k$-gon.*

The lower bound in the last part of the last theorem is the conjecture.

**Conjecture 2.5** *For all $k \geq 3$ for every set of $2^{k-2}+1$ points in $\mathbb{R}^2$ there exists $k$ of them that form a $k$-gon.*