Some Solutions to Midterm Problems

William Gasarch-U of MD
Problem 2

Prove the following and fill in the $f(k)$.

**Thm** For all $k$ there exists $n = f(k)$ such that the following holds. For all pairs of colorings:

$\text{COL}_1 : ([n]_1) \rightarrow [2],$

$\text{COL}_2 : ([n]_2) \rightarrow [2]$

$(\exists H \subseteq [n])(\exists c_1, c_2 \in \{1, 2\})$ such that
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- $H$ is of size $k,$
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**Thm** For all $k$ there exists $n = f(k)$ such that the following holds. For all pairs of colorings:

- $\text{COL}_1 : \binom{[n]}{1} \rightarrow [2],$
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$(\exists H \subseteq [n])(\exists c_1, c_2 \in \{1, 2\})$ such that

- $H$ is of size $k$,
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For all pairs of colorings:

- $\text{COL}_1 : ([n]^1) \rightarrow [2]$,
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$(\exists H \subseteq [n])(\exists c_1, c_2 \in \{1, 2\})$ such that

- $H$ is of size $k$,
- every element of $H$ is colored $c_1$, and
- every element of $^{n \choose 2}$ is colored $c_2$. 

Problem 2 Solution

\[ \text{COL}_1 : \binom{[n]}{1} \rightarrow [2], \]
\[ \text{COL}_2 : \binom{[n]}{2} \rightarrow [2]. \] We do the following.
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We determine $n$ later.

By 1-ary Ramsey ($\exists H_1)[|H_1| \geq \frac{n}{2}]$, $\text{COL}_1$ on $H_1$ is color $c_1$. We turn this around:

$$\forall \text{COL}_1: \binom{m}{2} \to [2] \exists H \text{ Homog} \ |H| \geq 0.5 \log_2(m).$$

Restrict $\text{COL}_2$ to $(H_1^2)$. Get:

$$|H| \geq 0.5 \log_2(|H_1|) = 0.5 \log_2(n^2) = 0.5 n^2.$$
Problem 2 Solution

\begin{align*}
\text{COL}_1 : (\binom{n}{1}) &\to [2], \\
\text{COL}_2 : (\binom{n}{2}) &\to [2]. \text{ We do the following.}
\end{align*}

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We apply 2-ary Ramsey. We showed in class:

\[
(\forall \text{COL} : \binom{2^{2k}}{2} \rightarrow [2])(\exists H)[H \text{ Homog } |H| \geq k].
\]
Problem 2 Solution

COL₁: \([n]_1 \mapsto [2]\),

COL₂: \([n]_2 \mapsto [2]\). We do the following.

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By 1-ary Ramsey \((\exists H₁)[|H₁| \geq \frac{n}{2}], \) COL₁ on \(H₁\) is color \(c₁\).

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\[(\forall \text{COL} : \binom{2^{2k}}{2} \mapsto [2])(\exists H)[H \text{ Homog } |H| \geq k].\]

We turn this around:
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\[ \text{COL}_1 : \binom{[n]}{1} \to [2], \]
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By 1-ary Ramsey \((\exists H_1)[|H_1| \geq n/2]\), \( \text{COL}_1 \) on \( H_1 \) is color \( c_1 \).

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(\forall \text{COL} : \binom{2^k}{2} \to [2])(\exists H)[H \text{ Homog } |H| \geq k].
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We turn this around:

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(\forall \text{COL} : \binom{m}{2} \to [2])(\exists H)[H \text{ Homog } |H| \geq 0.5 \log_2(m)].
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$\text{COL}_1: ([n]_1) \to [2],$

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Restrict $\text{COL}_2$ to $\binom{H_1}{2}$. Get: $|H| \geq 0.5 \log_2(|H_1|) = 0.5 \log_2\left(\frac{n}{2}\right)$. 
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\( \text{COL}_1 : ([n]_1) \rightarrow [2], \)
\( \text{COL}_2 : ([n]_2) \rightarrow [2]. \) We do the following.

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(\forall \text{COL} : \binom{m}{2} \rightarrow [2])(\exists H)[H \text{ Homog } |H| \geq 0.5 \log_2(m)].
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Restrict \( \text{COL}_2 \) to \( \binom{H_1}{2} \). Get: \(|H| \geq 0.5 \log_2(|H_1|) = 0.5 \log_2(\frac{n}{2}). \)

Need \( 0.5 \log_2(\frac{n}{2}) \geq k. \) Take \( n = 2^{2k+1}. \)
Problem 2. Why Important?

In the lang of graphs \((E(x,y))\) the question:

\[
\text{Given an } E^* A^* \text{ statement } \phi, \text{ find } \text{spec}(\phi)
\]

is decidable. And \(\text{spec}(\phi)\) is always finite or cofinite.

Key: Make the set \(Y\) very homog by making every element in \(Y\) have the same relation to every \(u \in U\) and to each other.

What if we added a unary predicate to the lang? So every element is colored RED or BLUE. Then we would need to also make every element of \(Y\) the same color.

This problem showed that YES we can do BOTH—make every element of \(Y\) the same color AND make every pair of elements of \(Y\) the same color.
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Problem 2. Why Important?

By iterating Ramsey we get the following theorem.
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the question: 

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And \(\text{spec}(\phi)\) is always finite or cofinite.
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*Given an \(E^*A^*\) statement \(\phi\), find \(\text{spec}(\phi)\) is decidable.*

And \(\text{spec}(\phi)\) is always finite or cofinite.

This is what Ramsey proved in his paper.
Problem 3

Let $T$ be the set of trees and $\leq$ be the minor ordering. Show that $(T, \leq)$ is a wqo.
Let $T$ be the set of trees and $\preceq$ be the minor ordering. Show that $(T, \preceq)$ is a wqo.

You may use any theorem that was PROVEN in class or on the HW. (Note that we DID NOT prove the Graph Minor Theorem, so you can’t use that.)
Problem 3. Form a Minimal Bad Sequence

Assume that there exists a bad seq.
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Let $T_1$ be the smallest tree that begins a bad seq. KILL.

Let $T_1, T_2, \ldots$ is called a minimal bad seq. 

Let $X = \bigcup_{i=1}^{\infty} \{T_{i1}, \ldots, T_{ik_i}\}$
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\text{\ldots}
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$(\forall i)$ take $T_i$ and rm root to get finite set of trees $T_{i_1}, \ldots, T_{i_k}$. 
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Assume not. Then $\exists$ bad seq.
Problem 3. $X$ is a WQO

\[ X = \bigcup_{i=1}^{\infty} \{ T_{i1}, \ldots, T_{ik_i} \} \]

Assume not. Then $\exists$ bad seq. Say it begins $T_{i_1j_1}$. We can assume $i_1$ is the smallest number that appears as a 1st index. $T_{i_1j_1}, T_{i_2j_2}, \ldots$ (we have $i_1 \leq i_2, i_3, \ldots$) we PREPEND $T_1, \ldots, T_{i_1-1}$ to the seq to get $T_1, T_2, \ldots, T_{i_1-1}, T_{i_1j_1}, T_{i_2j_2}, \ldots$ (we have $i_1 \leq i_2, i_3, \ldots$) For the rest goto the next slide.
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$$T_{i_1 j_1}, T_{i_2 j_2}, \ldots \text{ (We have } i_1 \leq i_2, i_3, \ldots)$$
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\[ T_{i_1j_1}, T_{i_2j_2}, \ldots \ (\text{We have } i_1 \leq i_2, i_3, \ldots) \]

we PREPEND $T_1, \ldots, T_{i_1-1}$ to the seq to get

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For the rest goto the next slide.
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$T_1, T_2, \ldots, T_{i_1-1}, T_{i_1j_1}, T_{i_2j_2}, \ldots (i_1 \leq i_2, i_3, \ldots)$

Claim This is a bad seq.
Problem 3. $X$ is a WQO

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\]

**Claim** This is a **bad seq**.

a) NO uptick within $T_1, \ldots, T_{i_1-1}$ since $T_1, T_2, \ldots$ is Bad Seq.
Problem 3. $X$ is a WQO

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b) NO uptick within $T_{i_1j_1}, \ldots$ since its a bad seq.
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a) NO uptick within $T_1, \ldots, T_{i_1-1}$ since $T_1, T_2, \ldots$ is Bad Seq.
b) NO uptick within $T_{i_1j_1}, \ldots$ since its a bad seq.
c) NO uptick $T_i \preceq T_{i_kj_k}$ since otherwise $T_i \preceq T_{i_k}$ and $i < i_k$. 
Problem 3. $X$ is a WQO

\[(\ast) \quad T_{i_1j_1}, T_{i_2j_2}, \ldots (i_1 \leq i_2, i_3, \ldots)\]

we PREPEND $T_1, \ldots, T_{i_1-1}$ to the seq to get

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**Claim** This is a **bad seq**.

a) NO uptick within $T_1, \ldots, T_{i_1-1}$ since $T_1, T_2, \ldots$ is Bad Seq.
b) NO uptick within $T_{i_1j_1}, \ldots$ since its a bad seq.
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**End of Proof of Claim**
**Problem 3.** \( X \) is a WQO

\[(*) \quad T_{i_1j_1}, T_{i_2j_2}, \ldots \quad (i_1 \leq i_2, i_3, \ldots)\]

we PREPEND \( T_1, \ldots, T_{i_1-1} \) to the seq to get

\[T_1, T_2, \ldots, T_{i_1-1}, T_{i_1j_1}, T_{i_2j_2}, \ldots \quad (i_1 \leq i_2, i_3, \ldots)\]

**Claim** This is a *bad seq*.

a) NO uptick within \( T_1, \ldots, T_{i_1-1} \) since \( T_1, T_2, \ldots \) is Bad Seq.
b) NO uptick within \( T_{i_1j_1}, \ldots \) since its a bad seq.
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**End of Proof of Claim**

\((*)\) is a bad seq that begins \( T_{i_1}, \ldots, T_{i_1-1} \) and then has \( T_{i_1j_1} \).
Problem 3. $X$ is a WQO

\[(*) \quad T_{i_1 j_1}, T_{i_2 j_2}, \ldots \ (i_1 \leq i_2, i_3, \ldots)\]

we PREPEND $T_1, \ldots, T_{i_1-1}$ to the seq to get

$$T_1, T_2, \ldots, T_{i_1-1}, T_{i_1 j_1}, T_{i_2 j_2}, \ldots \ (i_1 \leq i_2, i_3, \ldots)$$

Claim This is a bad seq.
a) NO uptick within $T_1, \ldots, T_{i_1-1}$ since $T_1, T_2, \ldots$ is Bad Seq.
b) NO uptick within $T_{i_1 j_1}, \ldots$ since its a bad seq.
c) NO uptick $T_i \preceq T_{i_k j_k}$ since otherwise $T_i \preceq T_{i_k}$ and $i < i_k$.

End of Proof of Claim

\[(*)\) is a bad seq that begins $T_{i_1}, \ldots, T_{i_1-1}$ and then has $T_{i_1 j_1}$. $T_{i_1}$ is the smallest tree that is right after $T_1, \ldots, T_{i_1-1}$ in a bad seq.
Problem 3. $X$ is a WQO

\[(\ast) \quad T_{i_1 j_1}, T_{i_2 j_2}, \ldots (i_1 \leq i_2, i_3, \ldots)\]

we PREPEND $T_1, \ldots, T_{i_1-1}$ to the seq to get

$T_1, T_2, \ldots, T_{i_1-1}, T_{i_1 j_1}, T_{i_2 j_2}, \ldots (i_1 \leq i_2, i_3, \ldots)$

**Claim** This is a bad seq.

a) NO uptick within $T_1, \ldots, T_{i_1-1}$ since $T_1, T_2, \ldots$ is Bad Seq.

b) NO uptick within $T_{i_1 j_1}, \ldots$ since its a bad seq.

c) NO uptick $T_i \preceq T_{i_k j_k}$ since otherwise $T_i \preceq T_{i_k}$ and $i < i_k$.

**End of Proof of Claim**

$(\ast)$ is a bad seq that begins $T_{i_1}, \ldots, T_{i_1-1}$ and then has $T_{i_1 j_1}$. $T_{i_1}$ is the smallest tree that is right after $T_1, \ldots, T_{i_1-1}$ in a bad seq.

$T_{i_1 j_1}$ is smaller than $T_{i_1}$, so contradiction.
Problem 3. $X$ is a WQO

\begin{align*}
(*) \quad & T_{i_1j_1}, T_{i_2j_2}, \ldots \ (i_1 \leq i_2, i_3, \ldots) \\
\text{we PREPEND } & T_1, \ldots, T_{i_1-1} \text{ to the seq to get} \\
& T_1, T_2, \ldots, T_{i_1-1}, T_{i_1j_1}, T_{i_2j_2}, \ldots \ (i_1 \leq i_2, i_3, \ldots)
\end{align*}

Claim This is a bad seq.
a) NO uptick within $T_1, \ldots, T_{i_1-1}$ since $T_1, T_2, \ldots$ is Bad Seq.
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End of Proof of Claim

$(*$) is a bad seq that begins $T_{i_1}, \ldots, T_{i_1-1}$ and then has $T_{i_1j_1}$.
$T_{i_1}$ is the smallest tree that is right after $T_1, \ldots, T_{i_1-1}$ in a bad seq.
$T_{i_1j_1}$ is smaller than $T_{i_1}$, so contradiction.

End of proof that $X$ is wqo
Problem 3. $X$ is wqo so by HW...

Recall HW04
Problem 3. $X$ is wqo so by HW. . .

Recall HW04

Assume $(X, \preceq)$ is a wqo.
Problem 3. $X$ is wqo so by HW.

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Let $PF(X)$ be the set of finite subsets of $X$. 
Recall HW04

Assume \((X, \preceq)\) is a wqo.

Let \(\text{PF}(X)\) be the set of finite subsets of \(X\).

Let \(\preceq'\) be the following order on \(\text{PF}(X)\).
Recall HW04

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Let \(PF(X)\) be the set of finite subsets of \(X\).

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Let \(Y, Z \in PF(X)\).
Recall HW04

Assume \((X, \preceq)\) is a wqo.
Let \(PF(X)\) be the set of finite subsets of \(X\).
Let \(\preceq'\) be the following order on \(PF(X)\).
Let \(Y, Z \in PF(X)\).
\(Y \preceq' Z\) iff \((\exists\) injective \(f : Y \to Z)(\forall y \in Y)[y \preceq f(y)]\).
Problem 3. $X$ is wqo so by HW... 

Recall HW04

Assume $(X, \preceq)$ is a wqo.

Let $PF(X)$ be the set of finite subsets of $X$.

Let $\preceq'$ be the following order on $PF(X)$.

Let $Y, Z \in PF(X)$.

$Y \preceq' Z$ iff $(\exists$ injective $f : Y \to Z)(\forall y \in Y)[y \preceq f(y)]$.

Then $(PF(X), \preceq')$ is a wqo.
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Assume \((X, \preceq)\) is a wqo.
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Let \(Y, Z \in PF(X)\).
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Then \((PF(X), \preceq')\) is a wqo.
We will use this.
Problem 3. View the Min Bad Seq As...

The Original Min Bad Sequence is

\[ T_1, T_2, \ldots \]
Problem 3. View the Min Bad Seq As... 

The Original Min Bad Sequence is 

$$T_1, T_2, \ldots$$

View this as a seq of finite sets of trees from wqo $X$. 

$$\{T_{11}, \ldots, T_{1k_1}\}, \{T_{21}, \ldots, T_{2k_2}\}, \ldots$$
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\( T_{i_1} \) is a minor of SOME elt of \( \{ T_{j_1}, \ldots, T_{jk_j} \} \).
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\[ \vdots \]
\( T_{ik_i} \) is a minor of SOME other elt of \( \{ T_{j1}, \ldots, T_{jk_j} \} \).

You can put all this together to get \( T_i \) is a minor of \( T_j \), which
contradicts \( T_1, \ldots, \) being a bad seq.
Problem 3: Afterthought

What did we use about \textbf{minor} in the proof?
Problem 3: Afterthought

What did we use about *minor* in the proof?

Would the same proof show that the subgraph-ordering for trees is a wqo?
Problem 3: Afterthought

What did we use about minor in the proof?

Would the same proof show that the subgraph-ordering for trees is a wqo?

I leave this for you to ponder.
Problem 3: Another Afterthought

Let $G$ be the set of all graphs and $\leq$ be the subgraph ordering.
Let $\mathcal{G}$ be the set of all graphs and $\preceq$ be the subgraph ordering.

**Vote**
Problem 3: Another Afterthought

Let $\mathcal{G}$ be the set of all graphs and $\preceq$ be the subgraph ordering.

Vote

a) $(\mathcal{G}, \preceq)$ is a wqo and this is known.
Problem 3: Another Afterthought

Let \( \mathcal{G} \) be the set of all graphs and \( \preceq \) be the subgraph ordering.

Vote

a) \( (\mathcal{G}, \preceq) \) is a wqo and this is known.

a) \( (\mathcal{G}, \preceq) \) is not a wqo and this is known.
Problem 3: Another Afterthought

Let $\mathcal{G}$ be the set of all graphs and $\preceq$ be the subgraph ordering.

**Vote**

a) $(\mathcal{G}, \preceq)$ is a wqo and this is known.

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a) The question “is $(\mathcal{G}, \preceq)$ a wqo?” is **unknown to science**.
Let $\mathcal{G}$ be the set of all graphs and $\preceq$ be the subgraph ordering.

**Vote**

a) $(\mathcal{G}, \preceq)$ is a wqo and this is known.
a) $(\mathcal{G}, \preceq)$ is not a wqo and this is known.
c) The question “is $(\mathcal{G}, \preceq)$ a wqo?” is unknown to science.

Answer on next slide.
Graphs under Subgraph

Let $C_i$ be the cycle on $i$ vertices. $C_3, C_4, C_5, ...$ is an infinite seq of incomparable elements, so graphs under subgraph are NOT a wqo.
Graphs under Subgraph

Let $C_i$ be the cycle on $i$ vertices.

$C_3, C_4, C_5, \ldots$

is an infinite seq of incomparable elements, so graphs under subgraph are NOT a wqo.
Problem 4

Prove or Disprove:

For every $\text{COL}: \mathbb{Q} \to [100]$ there exists an $H \subseteq \mathbb{Q}$ such that

- $H$ has the same order type as the rationals:

  - $H$ is countable
  - $H$ is dense: $(\forall x, y \in H) [x < y \Rightarrow (\exists z) x < z < y]$.
  - $H$ has no left endpoint: $(\forall y \in H) (\exists x \in H) x < y$.
  - $H$ has no right endpoint: $(\forall x \in H) (\exists y \in H) x < y$.

- every number in $H$ is the same color.

TRUE. We prove it TWO ways.
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$\quad \nabla \ H \text{ has the same order type as the rationals:}$

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Advice

You should understand both proofs.
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   d) $H$ has no right endpoint: $(\forall x \in H)(\exists y \in H)[x < y].$
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Advice You should understand both proofs.
We Actually Prove

**Def** Let $L$ be a linear ordering.

a) $L \equiv Q$ means $L$ has same order type as $Q$. Hence $L$ is countable, dense, and has no endpoints.
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a) $L \equiv Q$ means $L$ has same order type as $Q$. Hence $L$ is countable, dense, and has no endpoints.

b) Let $\text{COL}: L \rightarrow [c]$. $H$ is $\textbf{Q-homog}$ if $H$ is homog & $H \equiv Q$.

We use $c$ instead of 100 since we can then do an induction on $c$.

We use $L$ instead of $Q$ since in the induction proof we will have a coloring of (say) $(a, b)$ and want to use the Ind Hyp on a $\text{COL}$ restricted to $(a, b)$. 
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We will prove the following:

$$(\forall c)(\forall L \equiv Q)(\forall \text{COL}: L \rightarrow [c])(\exists H \subseteq L)[H \text{ Q-homog}].$$
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**Def** Let $L$ be a linear ordering.

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\[(\forall c)(\forall \text{COL}: L \to [c])(\exists H \subseteq L) H \text{ is Q-homog}\]

**Proof One and Proof Two Begin the Same Way**

We prove this by induction on \(c\).
\[(\forall c)(\forall \text{COL}: L \rightarrow [c])(\exists H \subseteq L)H \text{ is Q-homog}\]

Proof One and Proof Two Begin the Same Way

We prove this by induction on \(c\).

**IB** \(c = 1\). Obviously true.
Proof One and Proof Two Begin the Same Way
We prove this by induction on $c$.

**IB** $c = 1$. Obviously true.

**IH** Assume true for $c - 1$. 

$(\forall c)(\forall \text{COL}: L \rightarrow [c])(\exists H \subseteq L)H$ is Q-homog
(∀c)(∀COL: L → [c])(∃H ⊆ L)H is Q-homog

Proof One and Proof Two Begin the Same Way
We prove this by induction on c.
IB c = 1. Obviously true.
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Continued on Next Slide.
Induction Step for Proof One.

Let \( \text{COL} : L \rightarrow [c] \).
Induction Step for Proof One.

Let \( \text{COL} : L \rightarrow [c] \).
Let

\[ H = \{ x \in L : \text{COL}(x) = c \}. \]
Induction Step for Proof One.

Let \( \text{COL} : L \rightarrow [c] \).

Let \( H = \{ x \in L : \text{COL}(x) = c \} \).

**Case 1** \( H \equiv Q \). DONE!
Induction Step for Proof One.

Let \( \text{COL} : L \rightarrow [c] \).

Let

\[
H = \{ x \in L : \text{COL}(x) = c \}.
\]

**Case 1** \( H \equiv Q \). DONE!

**Case 2** \( H \not\equiv Q \). Three possibilities.

Case 2a \( H \) is not dense. So \( \exists x < y \in H \) such that \( (x, y) \cap H = \emptyset \). Nothing in \( (x, y) \) is colored \( c \).

Let \( \text{COL}' \) be \( \text{COL} \) restricted to \( (x, y) \). This is a \( c-1 \) coloring on \( (x, y) \) \( \equiv Q \). Done by IH.

Case 2b \( H \) has a left endpoint. So \( \exists y \) such that \( (-\infty, y) \cap H = \emptyset \). Let \( x \in L \) such that \( x < y \). Let \( \text{COL}' \) be \( \text{COL} \) restricted to \( (x, y) \). This is a \( c-1 \) coloring on \( (x, y) \) \( \equiv Q \). Done by IH.

Case 2c \( H \) has a right endpoint. Similar to Case 2b.

End of Proof One.
Induction Step for Proof One.

Let COL: \( L \to [c] \).

Let

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H = \{ x \in L : \text{COL}(x) = c \}.
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Induction Step for Proof One.

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End of Proof One
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Nothing in $(x, y)$ is colored $c$.
Let $\text{COL}'$ be $\text{COL}$ restricted to $(x, y)$.
This is a $c - 1$ coloring on $(x, y) \equiv Q$. Done by IH.

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Case 2a \( H \) is not dense. So \( (\exists x < y \in H)[(x, y) \cap H = \emptyset] \).
Nothing in \( (x, y) \) is colored \( c \).
Let \( \text{COL}' \) be \( \text{COL} \) restricted to \( (x, y) \).
This is a \( c - 1 \) coloring on \( (x, y) \equiv Q \). Done by IH.

Case 2b \( H \) has a left endpoint. So \( (\exists y)[(-\infty, y) \cap H = \emptyset] \). Let \( x \in L \) such that \( x < y \). Let \( \text{COL}' \) be \( \text{COL} \) restricted to \( (x, y) \).
This is a \( c - 1 \) coloring on \( (x, y) \equiv Q \). Done by IH.

Case 2c \( H \) has a right endpoint. Similar to Case 2b.
Induction Step for Proof One.

Let \( \text{COL} : L \rightarrow [c] \).
Let

\[
H = \{ x \in L : \text{COL}(x) = c \}.
\]

**Case 1** \( H \equiv Q \). DONE!

**Case 2** \( H \not\equiv Q \). Three possibilities.

**Case 2a** \( H \) is not dense. So \( (\exists x < y \in H)[(x, y) \cap H = \emptyset] \).
Nothing in \((x, y)\) is colored \( c \).
Let \( \text{COL}' \) be \( \text{COL} \) restricted to \((x, y)\).
This is a \( c-1 \) coloring on \((x, y) \equiv Q\). Done by IH.

**Case 2b** \( H \) has a left endpoint. So \( (\exists y)[(-\infty, y) \cap H = \emptyset] \). Let \( x \in L \) such that \( x < y \). Let \( \text{COL}' \) be \( \text{COL} \) restricted to \((x, y)\).
This is a \( c-1 \) coloring on \((x, y) \equiv Q\). Done by IH.

**Case 2c** \( H \) has a right endpoint. Similar to Case 2b.

End of Proof One
Induction Step for Proof Two: Plan

We will try to **construct** a Q-homog set.

► We succeed! YEAH!
Induction Step for Proof Two: Plan

We will try to \textbf{construct} a Q-homog set.

- We succeed! YEAH!
- We fail! Then we will have an open interval \((x, y)\) where \textsc{COL} is never color \(c\). Use IH.
Induction Step for Proof Two: Action

Let $\text{COL} : L \to [c]$. 
Induction Step for Proof Two: Action

Let $\text{COL}: L \to [c]$.

We define a seq $q_1, q_2, \ldots$ such that $\{q_1, q_2, \ldots\}$ is Q-homog OR we fail.
Induction Step for Proof Two: Action

Let $\text{COL} : L \rightarrow [c]$.

We define a seq $q_1, q_2, \ldots$ such that $\{q_1, q_2, \ldots\}$ is $Q$-homog OR we fail.

Let $q_1 \in L$ such that $\text{COL}(q_1) = c$. (If no such exists, use IH.)
Induction Step for Proof Two: Action

Let \( \text{COL}: L \rightarrow [c] \).
We define a seq \( q_1, q_2, \ldots \) such that \( \{q_1, q_2, \ldots\} \) is Q-homog OR we fail.
Let \( q_1 \in L \) such that \( \text{COL}(q_1) = c \). (If no such exists, use IH.)
Assume \( q_1, \ldots, q_n \) have been defined and are all color \( c \). Order them to get \( p_1 < \cdots < p_n \).
Induction Step for Proof Two: Action

Let \( \text{COL}: L \to [c] \).
We define a seq \( q_1, q_2, \ldots \) such that \( \{q_1, q_2, \ldots\} \) is Q-homog OR we fail.
Let \( q_1 \in L \) such that \( \text{COL}(q_1) = c \). (If no such exists, use IH.)
Assume \( q_1, \ldots, q_n \) have been defined and are all color \( c \). Order them to get \( p_1 < \cdots < p_n \).

- If \( (\exists q < p_1)[\text{COL}(q) = c] \) then let \( q_{n+1} \) be \( q \).
Induction Step for Proof Two: Action

Let \( \text{COL} : L \rightarrow [c] \).
We define a seq \( q_1, q_2, \ldots \) such that \( \{q_1, q_2, \ldots\} \) is Q-homog OR we fail.
Let \( q_1 \in L \) such that \( \text{COL}(q_1) = c \). (If no such exists, use IH.)
Assume \( q_1, \ldots, q_n \) have been defined and are all color \( c \). Order them to get \( p_1 < \cdots < p_n \).

- If \( (\exists q < p_1)[\text{COL}(q) = c] \) then let \( q_{n+1} \) be \( q \).
  If NOT then \( \text{COL} : (p_1 - \epsilon, p_1) \rightarrow [c - 1] \). STOP. Use IH.
- For \( 1 \leq i \leq n \)
Induction Step for Proof Two: Action

Let $\text{COL}: L \to [c]$. We define a seq $q_1, q_2, \ldots$ such that $\{q_1, q_2, \ldots\}$ is Q-homog OR we fail.

Let $q_1 \in L$ such that $\text{COL}(q_1) = c$. (If no such exists, use IH.)

Assume $q_1, \ldots, q_n$ have been defined and are all color $c$. Order them to get $p_1 < \cdots < p_n$.

- If $(\exists q < p_1)[\text{COL}(q) = c]$ then let $q_{n+1}$ be $q$. If NOT then $\text{COL}: (p_1 - \epsilon, p_1) \to [c - 1]$. STOP. Use IH.

- For $1 \leq i \leq n$
  - If $(\exists p_i < q < p_{i+1})[\text{COL}(q) = c]$ then let $q_{n+i+1}$ be $q$. 

Case 1 Const never stops. $\{q_1, q_2, \ldots\} \equiv \text{Q-homog}$. Done!

Case 2 Const stops. $\exists a < b, \text{COL}: (a, b) \to [c - 1]$. Use IH.
Induction Step for Proof Two: Action

Let \( \text{COL}: L \rightarrow [c] \).
We define a seq \( q_1, q_2, \ldots \) such that \( \{q_1, q_2, \ldots\} \) is Q-homog OR we fail.
Let \( q_1 \in L \) such that \( \text{COL}(q_1) = c \). (If no such exists, use IH.)
Assume \( q_1, \ldots, q_n \) have been defined and are all color \( c \). Order them to get \( p_1 < \cdots < p_n \).

- If \( (\exists q < p_1)[\text{COL}(q) = c] \) then let \( q_{n+1} \) be \( q \).
  If NOT then \( \text{COL}: (p_1 - \epsilon, p_1) \rightarrow [c - 1] \). STOP. Use IH.

- For \( 1 \leq i \leq n \)
  If \( (\exists p_i < q < p_{i+1})[\text{COL}(q) = c] \) then let \( q_{n+i+1} \) be \( q \).
  If NOT then \( \text{COL}: (p_i, p_{i+1}) \rightarrow [c - 1] \). STOP. Use IH.

- If \( (\exists p_1 < q)[\text{COL}(q) = c] \) then let \( q_{2n+2} \) be \( q \).
Induction Step for Proof Two: Action

Let \( \text{COL} : L \rightarrow [c] \).
We define a seq \( q_1, q_2, \ldots \) such that \( \{q_1, q_2, \ldots\} \) is Q-homog OR we fail.
Let \( q_1 \in L \) such that \( \text{COL}(q_1) = c \). (If no such exists, use IH.)
Assume \( q_1, \ldots, q_n \) have been defined and are all color \( c \). Order them to get \( p_1 < \cdots < p_n \).

- If \( (\exists q < p_1)[\text{COL}(q) = c] \) then let \( q_{n+1} \) be \( q \).
  If NOT then \( \text{COL} : (p_1 - \epsilon, p_1) \rightarrow [c - 1] \). STOP. Use IH.

- For \( 1 \leq i \leq n \)
  If \( (\exists p_i < q < p_{i+1})[\text{COL}(q) = c] \) then let \( q_{n+i+1} \) be \( q \).
  If NOT then \( \text{COL} : (p_i, p_{i+1}) \rightarrow [c - 1] \). STOP. Use IH.

- If \( (\exists p_1 < q)[\text{COL}(q) = c] \) then let \( q_{2n+2} \) be \( q \).
  If NOT then \( \text{COL} : (p_n, p_n + \epsilon) \rightarrow [c - 1] \). STOP. Use IH.

Case 1
Cons never stops.
\( \{q_1, q_2, \ldots\} \equiv Q \) & homog. Done!

Case 2
Cons stops.
\( \exists a < b, \text{COL} : (a, b) \rightarrow [c - 1] \). Use IH.
Induction Step for Proof Two: Action

Let \( \text{COL}: L \to [c] \).
We define a seq \( q_1, q_2, \ldots \) such that \( \{q_1, q_2, \ldots\} \) is Q-homog OR we fail.
Let \( q_1 \in L \) such that \( \text{COL}(q_1) = c \). (If no such exists, use IH.)
Assume \( q_1, \ldots, q_n \) have been defined and are all color \( c \). Order them to get \( p_1 < \cdots < p_n \).

- If \( (\exists q < p_1)[\text{COL}(q) = c] \) then let \( q_{n+1} \) be \( q \).
  If NOT then \( \text{COL}: (p_1 - \epsilon, p_1) \to [c - 1] \). STOP. Use IH.

- For \( 1 \leq i \leq n \)
  If \( (\exists p_i < q < p_{i+1})[\text{COL}(q) = c] \) then let \( q_{n+i+1} \) be \( q \).
  If NOT then \( \text{COL}: (p_i, p_{i+1}) \to [c - 1] \). STOP. Use IH.

- If \( (\exists p_1 < q)[\text{COL}(q) = c] \) then let \( q_{2n+2} \) be \( q \).
  If NOT then \( \text{COL}: (p_n, p_n + \epsilon) \to [c - 1] \). STOP. Use IH.

Case 1 Const never stops. \( \{q_1, q_2, \ldots\} \equiv Q & \text{homog}. \) Done!
Induction Step for Proof Two: Action

Let $\text{COL}: L \rightarrow [c]$. We define a seq $q_1, q_2, \ldots$ such that $\{q_1, q_2, \ldots\}$ is $Q$-homog OR we fail.

Let $q_1 \in L$ such that $\text{COL}(q_1) = c$. (If no such exists, use IH.) Assume $q_1, \ldots, q_n$ have been defined and are all color $c$. Order them to get $p_1 < \cdots < p_n$.

- If $(\exists q < p_1)[\text{COL}(q) = c]$ then let $q_{n+1}$ be $q$.
  - If NOT then $\text{COL}: (p_1 - \epsilon, p_1) \rightarrow [c - 1]$. STOP. Use IH.

- For $1 \leq i \leq n$
  - If $(\exists p_i < q < p_{i+1})[\text{COL}(q) = c]$ then let $q_{n+i+1}$ be $q$.
  - If NOT then $\text{COL}: (p_i, p_{i+1}) \rightarrow [c - 1]$. STOP. Use IH.

- If $(\exists p_1 < q)[\text{COL}(q) = c]$ then let $q_{2n+2}$ be $q$.
  - If NOT then $\text{COL}: (p_n, p_n + \epsilon) \rightarrow [c - 1]$. STOP. Use IH.

**Case 1** Const never stops. $\{q_1, q_2, \ldots\} \equiv Q \& \text{homog. Done!}$

**Case 2** Const stops. $\exists a < b$, $\text{COL}: (a, b) \rightarrow [c - 1]$. Use IH.