Some Solutions to Midterm Problems

William Gasarch-U of MD

Problem 2

Prove the following and fill in the $f(k)$.

Thm For all k there exists $n = f(k)$ such that the following holds.

For all pairs of colorings:

$\mathrm{COL}_1 \colon \binom{[n]}{1} \to [2]$,

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- every element of $\binom{H}{2}$ is colored c_2.

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Restrict COL_2 to $\binom{H_1}{2}$. Get: $|H| \ge 0.5 \log_2(|H_1|) = 0.5 \log_2(\frac{n}{2})$.
Need $0.5 \log_2(\frac{n}{2}) \ge k$. Take $n = 2^{2k+1}$.

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What if we added a unary predicate to the lang? So every element is colored RED or BLUE. Then we would need to **also** make every element of Y the same color.

This problem showed that YES we can do BOTH- make every element of Y the same color AND make every pair of elements of Y the same color.

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This is what Ramsey proved in his paper.

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You may use any theorem that was PROVEN in class or on the HW. (Note that we DID NOT prove the Graph Minor Theorem, so you can't use that.)

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For the rest goto the next slide.

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Recall HW04

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Assume $(X, \preceq)$ is a wqo.

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Let $\mathrm{PF}(X)$ be the set of finite subsets of X.

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We will use this.

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You can put all this together to get T_i is a minor of T_j, which contradicts $T_1, \ldots,$ being a bad seq.

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I leave this for you to ponder.

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Answer on next slide.

Graphs under Subgraph

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Let C_i be the cycle on i vertices.

$$C_3, C_4, C_5, \ldots$$

is an infinite seq of incomparable elements, so graphs under subgraph are NOT a wqo.

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Prove or Disprove:
For every $\mathrm{COL}\colon \mathsf{Q} \to [100]$ *there exists an* $H \subseteq \mathsf{Q}$ *such that*

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Advice You should understand both proofs.

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Def Let L be a linear ordering.
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We use L instead of Q since in the induction proof we will have a coloring of (say) (a, b) and want to use the Ind Hyp on a COL restricted to (a, b).

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Continued on Next Slide.

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End of Proof One

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We will try to **construct** a Q-homog set.

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- We fail! Then we will have an open interval (x, y) where COL is never color c. Use IH.

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  If NOT then $\mathrm{COL}\colon (p_i, p_{i+1}) \rightarrow [c-1]$. STOP. Use IH.
- If $(\exists p_1 < q)[\mathrm{COL}(q) = c]$ then let q_{2n+2} be q.
  If NOT then $\mathrm{COL}\colon (p_n, p_n + \epsilon) \rightarrow [c-1]$. STOP. Use IH.

Induction Step for Proof Two: Action

Let $\mathrm{COL}\colon L \to [c]$.
We define a seq $q_1, q_2, \ldots$ such that $\{q_1, q_2, \ldots\}$ is Q-homog OR we fail.
Let $q_1 \in L$ such that $\mathrm{COL}(q_1) = c$. (If no such exists, use IH.)
Assume $q_1, \ldots, q_n$ have been defined and are all color c. Order them to get $p_1 < \cdots < p_n$.

- If $(\exists q < p_1)[\mathrm{COL}(q) = c]$ then let q_{n+1} be q.
  If NOT then $\mathrm{COL}\colon (p_1 - \epsilon, p_1) \to [c-1]$. STOP. Use IH.
- For $1 \le i \le n$
  If $(\exists p_i < q < p_{i+1})[\mathrm{COL}(q) = c]$ then let q_{n+i+1} be q.
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Case 1 Const never stops. $\{q_1, q_2, \ldots\} \equiv$ Q & homog. Done!

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Case 1 Const never stops. $\{q_1, q_2, \ldots\} \equiv$ Q & homog. Done!

Case 2 Const stops . $\exists a < b$, $\mathrm{COL}\colon (a, b) \to [c-1]$. Use IH.