Schur's Thm + FLT(for $n = 4$) implies Primes Infinite

July 19, 2023

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3. All three of these proofs are harder than the usual proof

Background Needed

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Schur's Theorem

Thm $(\forall c)(\exists S = S(c))$ st for all c-colorings $\mathrm{COL}\colon [S] \to [c]$ there exists x, y, z monochormatic such that $x + y = z$.

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So let $S(c) = R(3; c)$ (homog set 3, colors c).

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In modern terminology:

$$(\forall n \geq 3)(\forall x, y, z \in \mathbb{N} - \{0\})[x^n + y^n \neq z^n].$$

This has come to be known as **Fermat's Last Theorem**.

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1) The 7th Dr. Who had a 5-line proof that uses Boolean Algebra.

2) The 11th Dr. Who gave **The real proof** to a group of geniuses to gain their trust. He later said that it was Fermat's original proof (possible but unlikely) but that Fermat didn't write it down since he died in a duel (not true). The writers of the show either confused Galois with Fermat or meant to say that Fermat died in a duel in a dual timeline.

More Fiction about Fermat's Last Theorem

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My guess is that Tobin wrote this limerick:

> *A challenge for many long ages*
> *Had baffled the savants and sages*
>
> *Yet at last came the light*
> *Seems that Fermat was right*
>
> *To the margin add 200 pages.*

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This violates FLT for $n = 4$.